§10.7 VECTOR FUNCTIONS & SPACE CURVES

Def: A $\boxed{\text{VECTOR-VALUED FUNCTION}}$ THAT SPITS OUT VECTORS.

(AS OPPOSED TO REAL-VALUED FUNCTIONS, WHICH SPIT OUT REAL #'S, SCALARS)

e.g. $\vec{r} : (0, \infty) \longrightarrow V_3$

$\vec{r} : t \longmapsto \langle t, \sqrt{t}, \ln(t) \rangle$

$$\vec{r}(t) = \langle \underbrace{t}_{x(t)}, \underbrace{\sqrt{t}}_{y(t)}, \underbrace{\ln(t)}_{z(t)} \rangle$$

COMPONENT FUNCTIONS

ex. WHAT IS DOMAIN OF $\vec{r}(t) = \left\langle \dfrac{1}{\sqrt{100-t}}, \dfrac{t^2+1}{t^2-1}, \ln(\pi + t) \right\rangle$

Def: LIMIT OF VECTOR-VALUED FUNCTION: IF $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ THEN

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{x \to a} x(t), \lim_{x \to a} y(t), \lim_{x \to a} z(t) \right\rangle,$$

PROVIDED THE LIMITS OF THE COMPONENT FUNCTIONS EXIST.

Def: A VECTOR VALUED FUNCTION $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ IS CONTINUOUS AT a IF $\displaystyle\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$

i.e. IF THE COMPONENT FUNCTIONS ARE CONTINUOUS AT a.

CONTINUOUS VECTOR-VALUED FUNCTIONS PRODUCE SPACE CURVES.

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$\hookrightarrow$ PARAMETRIC EQ'S : $x = x(t)$

$y = y(t)$

$z = z(t)$

GRAPH EXAMPLES ON COMPUTER

DERIVATIVES

Def: $$\vec{r}\,'(t) := \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \left\langle \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}, \lim_{h \to 0} \frac{z(t+h) - z(t)}{h} \right\rangle$$

$$= \langle x'(t), y'(t), z'(t) \rangle$$

PROVIDED THE LIMITS EXIST.

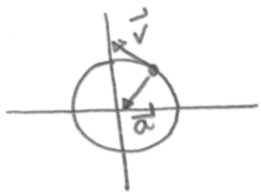
Def: $\vec{r}\,'(t_0)$ IS THE TANGENT VECTOR TO THE SPACE CURVE $\vec{r}(t)$ AT THE POINT $\vec{r}(t_0)$. THE TANGENT LINE TO $\vec{r}(t)$ AT $\vec{r}(t_0)$ IS THE LINE THROUGH $\vec{r}(t_0)$ $\parallel$ TO $\vec{r}\,'(t_0)$.

THE UNIT TANGENT VECTOR TO $\vec{r}(t)$ AT $\vec{r}(t_0)$ IS $\dfrac{\vec{r}\,'(t_0)}{|\vec{r}\,'(t_0)|}$.

REMARK: IF $\vec{r}(t)$ GIVES POSITION OF PARTICLE AT TIME t

THEN $\vec{r}\,'(t)$ & $\vec{r}\,''(t)$ GIVE VELOCITY & ACCELERATION VECTORS, RESPECTIVELY.

e.g. $\vec{r}(t) = \langle \cos t, \sin t \rangle$



THM: SUPPOSE $\vec{u}(t)$ & $\vec{v}(t)$ ARE DIFFERENTIABLE VECTOR-VALUED FUNCTIONS, c IS A SCALAR, AND $f(t)$ IS A REAL-VALUED FUNCTION. THEN

JUST LIMIT RULES!

(1) $\frac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}\,'(t) + \vec{v}\,'(t)$

(2) $\frac{d}{dt}\left[c\vec{u}(t)\right] = c\vec{u}\,'(t)$

* (3) $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$

(4) $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$

(5) $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$

PRODUCT RULES!

* (6) $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}\,'(f(t))$ CHAIN RULE!

* PROVE THESE

ex. SUPPOSE $|\vec{r}(t)| = c$ CONSTANT. THEN $\vec{r}\,'(t) \perp \vec{r}(t)$.

$$\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2$$

$$\Rightarrow \quad \frac{d}{dt}\left[\vec{r}(t) \cdot \vec{r}(t)\right] = 2\,\vec{r}\,'(t) \cdot \vec{r}(t) = 0$$

$$\vec{r}\,'(t) \cdot \vec{r}(t) = 0 \quad \text{ie. } \vec{r}\,'(t) \perp r(t).$$

COR. SUPPOSE $|\vec{r}\,'(t)| = c$ CONSTANT THEN $\vec{r}\,''(t) \perp \vec{r}\,'(t)$.

INTEGRALS

JUST LIKE DERIVATIVES, INTEGRALS OF VECTOR-VALUED FUNCTIONS ARE DEFINED COMPONENT-WISE: FOR CONTINUOUS $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$:

DEF: $$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt \,,\ \int_a^b y(t)\,dt \,,\ \int_a^b z(t)\,dt \right\rangle \leftarrow \text{VECTOR}$$

$$\int \vec{r}(t)\,dt = \left\langle \int x(t)\,dt \,,\ \int y(t)\,dt ,\ \int z(t)\,dt \right\rangle \leftarrow \text{ANTI-DERIVATIVE}$$

(VECTOR-VALUED FUNCTION)

(FOLLOWS FROM DISCUSSION OF HOW LIMITS ARE COMPUTED COMPONENT-WISE)

PROBLEMS:

- FIND UNIT TANGENT VECTOR $\vec{T}$ TO $\vec{r}(t) = \langle t^3 + 3t, t^2 + 1, 3t + 4 \rangle$ WHEN $t = 1$
- TANGENT LINE ?

- FIND A VECTOR EQ FOR THE TANGENT LINE TO THE INTERSECTION OF CYLINDERS

$$x^2 + y^2 = 25$$
$$y^2 + z^2 = 20$$