§ 10.7 VECTOR FUNCTIONS & SPACE CURVES

3. $$\lim_{t\to 0}\left(e^{-3t}\,\hat{i} + \frac{t^2}{\sin^2 t}\,\hat{j} + \cos(2t)\,\hat{k}\right)$$

$$= \left(\lim_{t\to 0} e^{-3t}\right)\hat{i} + \left(\lim_{t\to 0}\frac{t^2}{\sin^2 t}\right)\hat{j} + \left(\lim_{t\to 0}\cos(2t)\right)\hat{k}$$

$$= 1\,\hat{i} + \left(\lim_{t\to 0}\frac{2t}{2\sin(t)\cos(t)}\right)\hat{j} + 1\,\hat{k}$$

BY L'HÔPITAL'S RULE

$$= \boxed{\hat{i} + \hat{j} + \hat{k}}$$

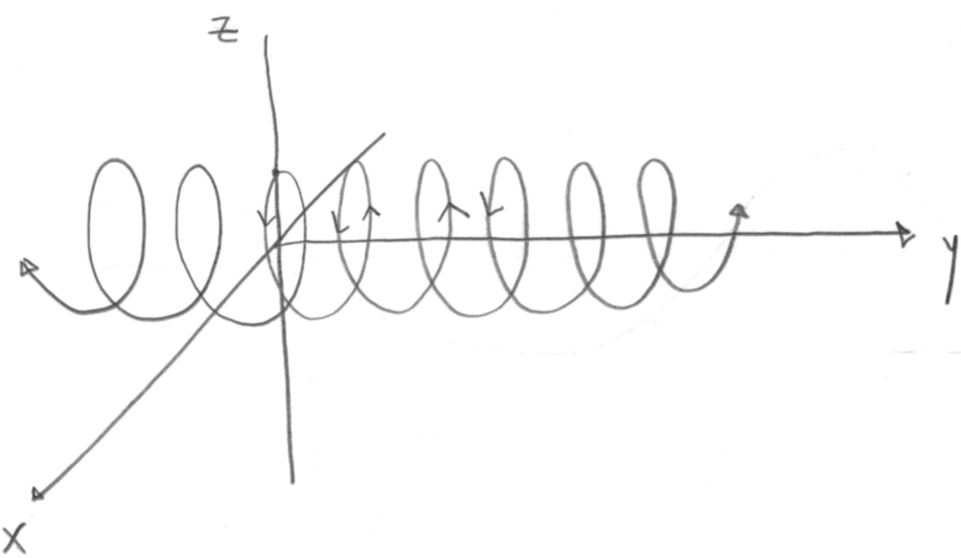
4. $$\lim_{t\to 1}\frac{t\cancel{(t-1)}}{\cancel{(t-1)}}\,\hat{i} + \lim_{t\to 1}\sqrt{t+8}\,\hat{j} + \lim_{t\to 1}\frac{\sin \pi t}{\ln t}\,\hat{k}$$

L'HOPITAL'S RULE

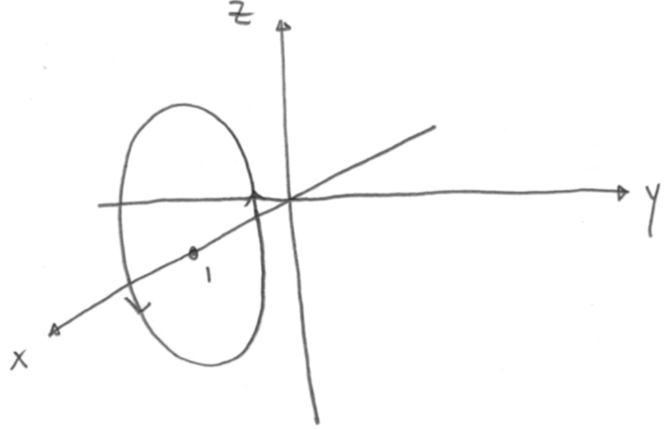
$$= 1\,\hat{i} + 3\,\hat{j} + \lim_{t\to 1}\frac{\pi\cos(\pi t)}{1/t}\,\hat{k}$$

$$= \boxed{\hat{i} + 3\hat{j} - \pi\hat{k}}$$

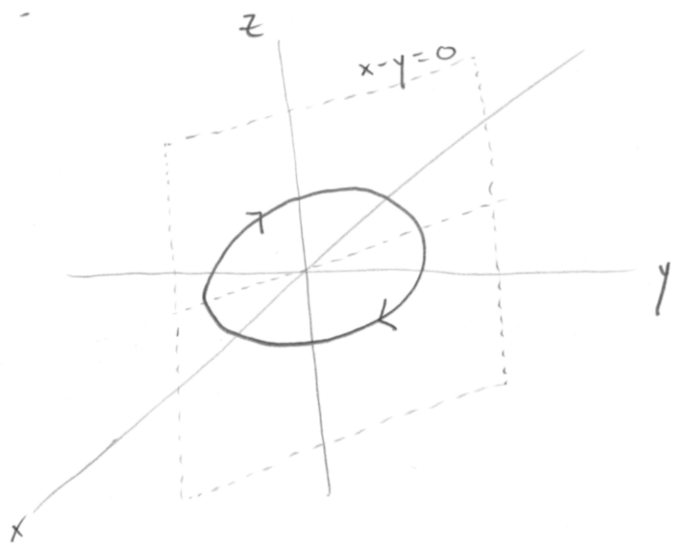
8.



9.



12. NOTE THAT $x(t) = -y(t)$ SO THIS CURVE LIES IN THE PLANE $x - y = 0$.



17. II $y = t$ & $x^2 + y^2 = t^2$

18. VI x & y COORD TRACE OUT CIRCLE,
z IS SMALL FOR LARGE $|t|$ BUT MAX VALUE 1 WHEN $t = 0$.

19. V x GROWS LINEARLY, y STAYS BOUNDED, z GROWS, ALWAYS POS.

20. I LIKE CIRCLE IN xy-PLANE, BUT z VALUE FLUCTUATES.

21. IV CURVE LIES ON CYLINDER $x^2 + y^2 = 1$, z GROWS EXPONENTIALLY

22. III CURVE LIES IN PLANE $x + y = 1$, z GROWS LINEARLY.

28. LET $x = 2\cos(t)$, $y = 2\sin(t)$ $\} \longleftarrow$ THIS COMES FROM $x^2 + y^2 = 4$

$z = (2\cos(t))(2\sin(t))$ $\longleftarrow$ THIS COMES FROM $z = xy$

$\therefore$ VECTOR FUNCTION : $\boxed{\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\cos t \sin t \rangle}$

29. BY TRANSITIVE PROP : $1 + y = \sqrt{x^2 + y^2}$ SQ. BOTH SIDES

$$1 + 2y + \cancel{y^2} = x^2 + \cancel{y^2} \longrightarrow 1 + 2y = x^2$$

$$y = \frac{x^2 - 1}{2}$$

JUST LET $x = t$. THEN $y = \dfrac{t^2 - 1}{2}$

$$z = 1 + y = 1 + \frac{t^2 - 1}{2}$$

$$= \frac{t^2 + 1}{2}$$

$$\therefore \boxed{\vec{r}(t) = \left\langle t, \frac{t^2 - 1}{2}, \frac{t^2 + 1}{2} \right\rangle}$$

42. $\vec{r}(t) = \langle at\cos(3t), b\sin^3(t), c\cos^3(t) \rangle$

$\vec{r}\,'(t) = \langle a\cos(3t) - 3at\sin(3t), 3b\sin^2(t)\cos(t), -3c\cos^2(t)\sin(t) \rangle$

43. $\vec{r}(t) = \vec{a} + t\vec{b} + t^2\vec{c}$

$$\boxed{\vec{r}\,'(t) = \vec{b} + 2t\vec{c}}$$

44. $\vec{r}(t) = t\vec{a} \times (\vec{b} + t\vec{c})$

$$\vec{r}\,'(t) = \frac{d}{dt}\left[t\vec{a}\right] \times (\vec{b} + t\vec{c}) + t\vec{a} \times \frac{d}{dt}(\vec{b} + t\vec{c})$$

$$= \vec{a} \times (\vec{b} + t\vec{c}) + t\vec{a} \times \vec{c}$$

$$= \vec{a} \times \vec{b} + t\vec{a} \times \vec{c} + t\vec{a} \times \vec{c} = \vec{a} \times \vec{b} + 2t\vec{a} \times \vec{b}$$

$$\boxed{\vec{r}\,'(t) = (1 + 2t)\,\vec{a} \times \vec{b}}$$

45. $\vec{r}(t) = \langle \cos t, 3t, 2\sin(2t) \rangle$

$\vec{r}\,'(t) = \langle -\sin t, 3, 4\cos(2t) \rangle$

$\vec{r}\,'(0) = \langle 0, 3, 4 \rangle \xrightarrow[\text{VECTOR}]{\text{UNIT}} \boxed{\left\langle 0, \frac{3}{5}, \frac{4}{5} \right\rangle}$

47. $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$$\vec{T}(1) = \frac{\vec{r}\,'(1)}{|\vec{r}\,'(1)|} = \frac{\langle 1, 2, 3 \rangle}{\sqrt{1^2 + 2^2 + 3^2}} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$$

$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2 \rangle$$

51. $\vec{r}(t) = \langle e^{-t}\cos t \,,\; e^{-t}\sin t \,,\; e^{-t} \rangle$

$\vec{r}\,'(t) = \langle -e^{-t}\cos t - e^{-t}\sin t \,,\; -e^{-t}\sin t + e^{-t}\cos t \,,\; -e^{-t} \rangle$

NOTE THAT $\vec{r}(0) = \langle 1, 0, 1 \rangle \leftarrow$ POINT $\vec{r}_0$

$\vec{r}\,'(1) = \langle -1, 1, -1 \rangle \leftarrow$ DIRECTION $\vec{v}$

LINE $\ell(t) = \vec{r}_0 + t\vec{v} = \langle 1, 0, 1 \rangle + t\langle -1, 1, -1 \rangle$

$$\boxed{\ell(t) = \langle 1-t \,,\; t \,,\; 1-t \rangle}$$

54. PLANE $\sqrt{3}\,x + y = 1$ HAS NORMAL VECTOR $\langle \sqrt{3}, 1, 0 \rangle$

SO FIND t SUCH THAT $\underbrace{\langle 2\cos t, 2\sin t, e^{t} \rangle \cdot \langle \sqrt{3}, 1, 0 \rangle = 0}$

NOTE: $\perp$ TO NORMAL VECTOR $\Longleftrightarrow \parallel$ TO PLANE

$$\rightarrow \quad 2\sqrt{3}\cos t + 2\sin t = 0$$

$$2\sin t = -2\sqrt{3}\cos t$$

$$\tan t = -\sqrt{3}$$

$$t = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3} \quad \left(\begin{matrix}\text{BECAUSE} \\ 0 \le t \le \pi\end{matrix}\right)$$

$\therefore$ POINT $\vec{r}\left(\frac{2\pi}{3}\right) = \left\langle 2\cos\left(\frac{2\pi}{3}\right), 2\sin\left(\frac{2\pi}{3}\right), e^{2\pi/3} \right\rangle$

$$= \boxed{\langle -1, \sqrt{3}, e^{2\pi/3} \rangle}$$

57. $\vec{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$, $\vec{r}_2'(t) = \langle \cos(t), 2\cos(2t), 1 \rangle$

OF COURSE, CURVES INTERSECT WHEN $t=0$.

$\vec{r}_1(0) = \langle 1, 0, 0 \rangle$, $\vec{r}_2(0) = \langle 1, 2, 1 \rangle$

$$\cos\theta = \frac{\vec{r}_1(0) \cdot \vec{r}_2(0)}{|\vec{r}_1(0)||\vec{r}_2(0)|} = \frac{1+0+0}{(1)(\sqrt{6})} = \frac{1}{\sqrt{6}}$$

$$\boxed{\theta = \cos^{-1}\frac{1}{\sqrt{6}}} \approx 65.91^\circ$$

59. $\left\langle \int_0^2 t\,dt , \int_0^2 -t^3\,dt , \int_0^2 3t^5\,dt \right\rangle$

$$= \left\langle \frac{1}{2}t^2 \Big|_0^2 , -\frac{1}{4}t^4 \Big|_0^2 , \frac{1}{2}t^6 \Big|_0^2 \right\rangle$$

$$= \boxed{\langle 2, -4, 32 \rangle}$$

61. $\left\langle \int_0^{\pi/2} 3\sin^2 t\cos t\,dt , \int_0^{\pi/2} 3\sin t\cos^2 t\,dt , \int_0^{\pi/2} 2\sin t\cos t\,dt \right\rangle$

$$= \left\langle 3\int_0^1 u^2\,du , -3\int_1^0 u^2\,dt , 2\int_0^1 u\,du \right\rangle$$

$$= \left\langle u^3 \Big|_0^1 , -u^3 \Big|_1^0 , u^2 \Big|_0^1 \right\rangle = \boxed{\langle 1, 1, 1 \rangle}$$

63. $\left\langle \int \sec^2 t \, dt, \int t(t^2+1)^3 \, dt, \int t^2 \ln t \, dt \right\rangle$

$= \left\langle \tan(t) + c_1, \frac{1}{8}(t^2+1)^4 + c_2, \frac{1}{3}t^3 \ln t - \int \frac{1}{3} t^2 \, dt \right\rangle$

$= \left\langle \tan(t) + c_1, \frac{1}{8}(t^2+1)^4 + c_2, \frac{1}{3}t^3 \ln(t) - \frac{1}{9} t^3 + c_3 \right\rangle$

or $\boxed{\left\langle \tan(t), \frac{1}{8}(t^2+1)^4, \frac{1}{3}t^3 \ln(t) - \frac{1}{9}t^3 \right\rangle + \vec{c}}$

65. $\vec{r}(t) = \left\langle \int 2t \, dt, \int 3t^2 \, dt, \int \sqrt{t} \, dt \right\rangle$

$\vec{r}(t) = \left\langle t^2 + c_1, t^3 + c_2, \frac{2}{3} t^{3/2} + c_3 \right\rangle$

$\vec{r}(1) = \left\langle 1 + c_1, 1 + c_2, \frac{2}{3} + c_3 \right\rangle = \langle 1, 1, 0 \rangle$

so $c_1 = 0$, $c_2 = 0$, $c_3 = -\frac{2}{3}$

$\therefore \boxed{\vec{r}(t) = \left\langle t^2, t^3, \frac{2}{3} t^{3/2} - \frac{2}{3} \right\rangle}$