§ 11.3 Partial Derivatives

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3. (a) Positive (b) Negative

7. $f(x,y) = y^5 - 3xy$

$$\boxed{f_x = -3y \quad , \quad f_y = 5y^4 - 3x}$$

9. $f(x,t) = e^{-t}\cos(\pi x)$

$$\boxed{f_x = -\pi e^{-t}\sin(\pi x) \quad , \quad f_t = -e^{-t}\cos(\pi x)}$$

13. $f(x,y) = \dfrac{ax+by}{cx+dy}$

$$f_x = \frac{(cx+dy)a - (ax+by)c}{(cx+dy)^2} = \boxed{\frac{(ad-bc)y}{(cx+dy)^2}}$$

$$f_y = \frac{(cx+dy)b - (ax+by)d}{(cx+dy)^2} = \boxed{\frac{(bc-ad)x}{(cx+dy)^2}}$$

17. $R(p,q) = \tan^{-1}(pq^2)$

$$\boxed{R_p = \frac{1}{1+(pq^2)^2}\cdot q^2 \quad , \quad R_q = \frac{1}{1+(pq^2)^2}\cdot 2pq}$$

Recall: $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$

18. $f(x,y) = x^y$

POWER RULE

RECALL:

$$\boxed{f_x = y x^{y-1}, \quad f_y = x^y \ln x}$$

$$\frac{d}{dx}\left[a^x\right] = a^x \ln a$$

19. $F(x,y) = \int_y^x \cos(e^t)\,dt$

$\boxed{F_x = \cos(e^x)}$ BY FUNDAMENTAL THM OF CALC

$$F_y = \frac{\partial}{\partial y}\int_y^x \cos(e^t)\,dt = \frac{\partial}{\partial y} - \int_x^y \cos(e^t)\,dt$$

$$= \boxed{-\cos(e^y)}$$

23. $w = \ln(x + 2y + 3z)$

$$\boxed{\frac{\partial w}{\partial x} = \frac{1}{x+2y+3z}, \quad \frac{\partial w}{\partial y} = \frac{2}{x+2y+3z}, \quad \frac{\partial w}{\partial z} = \frac{3}{x+2y+3z}}$$

26. $u = x^{y/z}$

$$\frac{\partial u}{\partial x} = \frac{y}{z} x^{\frac{y}{z}-1}, \quad \frac{\partial u}{\partial y} = \frac{1}{z} x^{y/z} \ln x,$$

$$\frac{\partial u}{\partial z} = -\frac{y}{z^2} x^{y/z} \ln x$$

29. $u = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

$$\boxed{\frac{\partial u}{\partial x_i} = \frac{\cancel{2} x_i}{\cancel{2}\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}}}$$

41. $e^z = xyz$

$\frac{\partial z}{\partial x}$: $e^z \cdot \frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x} \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}}$

$\frac{\partial z}{\partial y}$: $e^z \cdot \frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y} \Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}}$

42. $yz + x \operatorname{Ln} y = z^2$

$\frac{\partial z}{\partial x}$: $y\frac{\partial z}{\partial x} + \operatorname{Ln} y = 2z\frac{\partial z}{\partial x} \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{\operatorname{Ln} y}{2z - y}}$

$\frac{\partial z}{\partial y}$: $z + y\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y} \Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{z + \frac{x}{y}}{2z - y}}$

45. $f(x,y) = x^3y^5 + 2x^4y$

SAME SINCE THIS IS CONTINUOUS EVERYWHERE (POLYNOMIAL!) [CLAIRAUT'S THM]

$f_x = 3x^2y^5 + 8x^3y \longrightarrow \boxed{f_{yx} = f_{xy} = 15x^2y^4 + 8x^3}$

$\boxed{f_{xx} = 6xy^5 + 24x^2y}$

$f_y = 5x^3y^4 + 2x^4 \longrightarrow \boxed{f_{yy} = 20x^3y^3}$

59. $f(x,y,z) = xy^2z^3 + \sin^{-1}(x\sqrt{z})$

$f_y = 2xyz^3 \longrightarrow f_{zy} = 6xyz^2 \longrightarrow \boxed{f_{xzy} = 6yz^2}$

DEFINITELY EASIEST WHEN WE TAKE $\frac{\partial}{\partial y}$ FIRST

BECAUSE $\frac{\partial}{\partial y} \sin^{-1}(x\sqrt{z}) = 0$

60. $g(x,y,z) = \underbrace{\sqrt{1+xz}} + \underbrace{\sqrt{1-xy}}$

THIS TERM DOESN'T HAVE y'S
& NEITHER WILL ANY OF ITS
PARTIAL DERIVATIVES.

SIMILARLY, NO z'S.
WHEN WE EVENTUALLY TAKE
$\frac{\partial}{\partial z}$ WE WILL GET 0.

SO, NO MATTER WHEN WE TAKE $\frac{\partial}{\partial y}$
(FIRST, SECOND, OR THIRD) WE WILL GET 0.
MIGHT AS WELL DO IT FIRST!

$\therefore \boxed{g_{xyz} = 0}$

63. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$

$u_x = -\cancel{\tfrac{1}{2}}(x^2+y^2+z^2)^{-3/2} \cdot \cancel{2}x$

$u_{xx} = \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2} \cdot \cancel{2}x^2 - (x^2+y^2+z^2)^{-3/2}$

$= (x^2+y^2+z^2)^{-5/2}\left[3x^2 - (x^2+y^2+z^2)\right]$

$$u_{xx} = \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}}$$

SIMILARLY, $$u_{yy} = \frac{2y^2 - x^2 - z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$u_{zz} = \frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\therefore\ u_{xx} + u_{yy} + u_{zz} = \frac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

66. $u = e^{\underbrace{a_1x_1 + a_2x_2 + \dots + a_nx_n}}$, $\underbrace{a_1^2 + a_2^2 + \dots + a_n^2 = 1}$

i.e. $\vec{a} \cdot \vec{x}$ WHERE $\vec{a}, \vec{x}$ ARE n-DIM. VECTORS

i.e. $\vec{a}$ IS AN n-DIM. UNIT VECTOR.

THEN $$\frac{\partial u}{\partial x_i} = e^{a_1x_1 + a_2x_2 + \dots + a_nx_n} \cdot a_i$$

$$\frac{\partial^2 u}{\partial x_i^2} = e^{a_1x_1 + a_2x_2 + \dots + a_nx_n} \cdot a_i^2 = a_i^2 u$$

$$\therefore\ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = a_1^2 u + a_2^2 u + \dots + a_n^2 u$$

$$= (a_1^2 + a_2^2 + \dots + a_n^2) u = u$$

67. $z = xe^y + ye^x$

$\frac{\partial z}{\partial x} = e^y + ye^x \longrightarrow \underbrace{\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}}_{\text{CLAIRAUT'S THM}} = e^y + e^x$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^3 z}{\partial x^3} = ye^x$

$\frac{\partial^3 z}{\partial y^2 \partial x} = \frac{\partial^3 z}{\partial x \partial y^2} = e^y$

$\frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial^3 z}{\partial x^2 \partial y} = e^x$

$\frac{\partial z}{\partial y} = xe^y + e^x$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^3 z}{\partial y^3} = xe^y$

$\therefore \frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial y^3} = x \frac{\partial^3 z}{\partial x \partial y^2} + y \frac{\partial^3 z}{\partial y \partial x^2}$ BECOMES

$ye^x + xe^y = x(e^y) + y(e^x)$ ✓

69. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. TAKE $\frac{\partial}{\partial R_1}$ OF EVERYTHING.

$\frac{-1}{R^2} \frac{\partial R}{\partial R_1} = \frac{-1}{R_1^2} \Rightarrow \boxed{\frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}}$

75. IF $f_x = x + 4y$ THEN $f = \int (x + 4y)\,dx$

$$= \frac{1}{2}x^2 + 4xy + g(y)$$

NOTE THAT THE CONSTANT OF INTEGRATION IS NOW ANY FUNCTION OF y ONLY (WHICH COULD BE A CONSTANT) SINCE $\frac{\partial}{\partial x} g(y) = 0$.

& IF $f_y = 3x - y$ THEN $f = \int (3x - y)\,dy$

$$= 3xy - \frac{1}{2}y^2 + h(x)$$

BUT $\frac{1}{2}x^2 + 4xy + g(y) \neq 3xy - \frac{1}{2}y^2 + h(x)$

NO MATTER WHAT $g(y)$ & $h(x)$ ARE.

$\therefore$ NO, YOU SHOULD NOT BELIEVE IT.