§ 11.4 TANGENT PLANES & LINEAR APPROXIMATIONS

LAST TIME:

IF f HAS CONTINUOUS PARTIAL DERIVATIVES

THEN AN EQUATION FOR THE TANGENT PLANE

TO THE SURFACE $z = f(x,y)$ AT THE POINT

(x_0, y_0, z_0) IS

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(NOTE: $z_0 = f(x_0, y_0)$)

EX: FIND EQ OF TANGENT PLANE TO $z = 2x^2 + 3y^2 + 1$

AT PNT. $(1, 1, 6)$

ANS:

$f_x = 4x$	$f_y = 6y$
$f_x(1,1) = 4$	$f_y(1,1) = 6$

$$\rightarrow \quad z - 6 = 4(x-1) + 6(y-1)$$

$$z = 4x + 6y - 4$$

ex: FIND EQ OF LINE $\perp$ TO $z = \dfrac{x+1}{y-1}$ AT $(2,2,3)$.

1. TANGENT PLANE: $f_x = \dfrac{1}{y-1}$ | $f_y = -\dfrac{x+1}{(y-1)^2}$

$f_x(2,2) = 1$ | $f_y(2,2) = -3$

$$z - 3 = 1(x-2) - 3(y-2)$$

$$z = x - 3y + 7$$

OR $$x - 3y - z = -7$$

2. NORMAL VECTOR : $\langle 1, -3, -1 \rangle$

PNT : $(2, 2, 3)$

$$\vec{r}(t) = \langle 2, 2, 3 \rangle + t\langle 1, -3, -1 \rangle$$

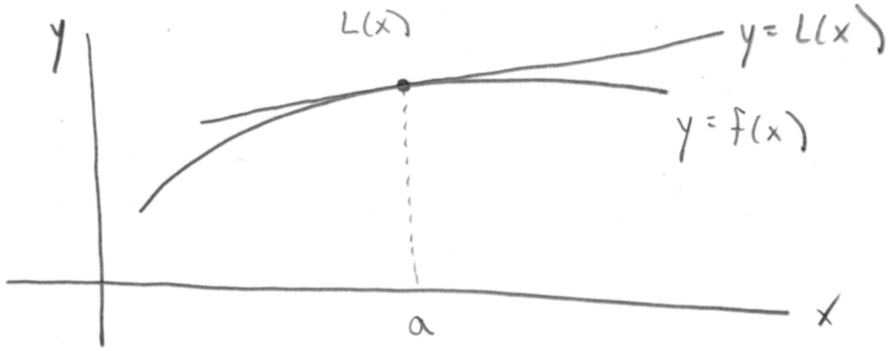
$$= \langle 2+t, 2-3t, 3-t \rangle$$

LINEAR APPROXIMATIONS

(THE TANGENT LINE)

RECALL 1-VARIABLE CASE: WE USE A LINEAR FUNCTION OF X TO APPROXIMATE A DIFFERENTIABLE FUNCTION OF X OVER SMALL DISTANCES.

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{L(x)} \quad \text{FOR } x \text{ NEAR } a.$$



ex: USE LINEAR APPROX TO ESTIMATE $\sqrt{16.1}$

$f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{16}}$

WE HAVE

$$f(x) \approx f(16) + f'(16)(x-16)$$

FOR x NEAR 16

$$f(16.1) \approx 4 + \frac{1}{8}(16.1 - 16)$$

$$= 4 + \frac{.1}{8} = 4.0125$$

2-VARIABLES: WE USE A LINEAR FUNCTION OF x & y (THE TANGENT PLANE) TO APPROXIMATE A DIFFERENTIABLE FUNCTION OF x & y OVER SMALL DISTANCES.

$$f(x,y) \approx \underbrace{f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}_{L(x,y)}$$

DIFFERENTIABILITY OF FUNCTIONS OF 2 VARIABLES

RECALL THAT A FUNCTION OF 1 VARIABLE IS DIFFERENTIABLE AT A POINT IF ITS DERIVATIVE EXISTS AT THAT POINT.

(THAT IS, f IS DIFFERENTIABLE AT a IF

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{EXISTS})$$

SO YOU MIGHT GUESS THAT A FUNCTION OF 2 VARIABLES IS DIFFERENTIABLE AT A POINT IF ITS 2 PARTIAL DERIVATIVES EXIST AT THAT POINT.

BUT THE TRUTH IS MORE COMPLICATED.

CONSIDER

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{IF } (x,y) \neq (0,0) \\ 0 & \text{IF } (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{1}{h} \, \frac{h \cdot 0}{h^2 + 0^2} = 0$$

& $f_y(0,0) = 0$ SIMILARLY.

BOTH PARTIAL DERIVATIVES EXIST.

SO ITS LINEAR APPROXIMATION WOULD BE

$$f(x,y) \approx f_x(0,0)(x-0) + f_y(0,0)(y-0) \quad \text{FOR } (x,y) \text{ NEAR } (0,0)$$

$$f(x,y) \approx 0$$

i.e. TANGENT PLANE $z = 0$.

HOWEVER, ON LINE $y = x$,

$$f(x,y) = f(x,x) = \frac{x^2}{x^2 + x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

SO WE CANNOT APPROXIMATE f WITH A LINEAR FUNCTION NEAR ORIGIN.

ITS GRAPH DOES NOT HAVE A TANGENT PLANE AT ORIGIN.

[NOTE THAT f HAD PARTIAL DERIVATIVES AT $(0,0)$ EVEN THOUGH IT WASN'T CONTINUOUS AT $(0,0)$! (DEFINED TO BE 0 THERE!)]

Def: $f(x,y)$ IS DIFFERENTIABLE AT (a,b) IF IT CAN BE APPROXIMATED BY A LINEAR FUNCTION NEAR (a,b)

DIFFERENTIABLE AT (a,b) $\Longleftrightarrow$ CAN BE APPROXIMATED BY LINEAR FUNCTION NEAR (a,b)

TANGENT PLANE APPROXIMATES THE GRAPH $z = f(x,y)$ WELL NEAR PNT. OF TANGENCY.

THM: IF THE PARTIAL DERIVATIVES f_x & f_y EXIST NEAR (a,b) **AND** ARE CONTINUOUS AT (a,b) THEN f IS DIFFERENTIABLE AT (a,b).

ex: EXPLAIN WHY $f(x,y) = \sqrt{x + e^{4y}}$ IS DIFFERENTIABLE AT $(3,0)$. THEN FIND ITS LINEARIZATION $L(x,y)$.

$$\underbrace{f_x = \frac{1}{2\sqrt{x+e^{4y}}} \qquad f_y = \frac{2e^{4y}}{\sqrt{x+e^{4y}}}}$$

CONTINUOUS OF DOMAIN. $(3,0)$ IS IN DOMAIN

$\therefore$ DIFFERENTIABLE AT $(3,0)$

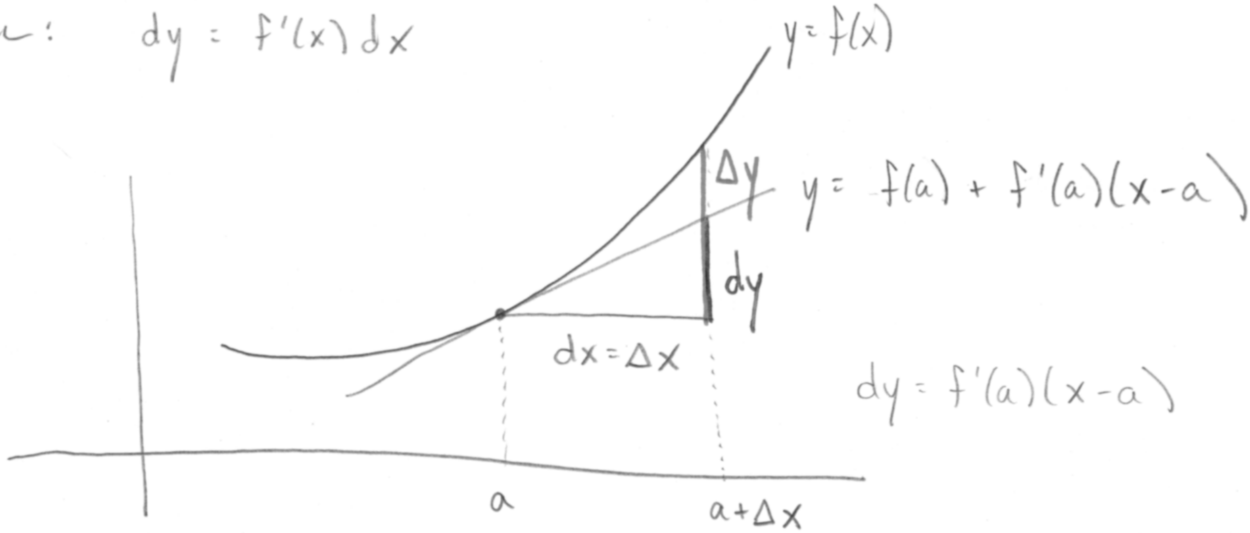
$$f_x(3,0) = \frac{1}{4} \qquad f_y(3,0) = 1$$

$$L(x,y) = f(3,0) + f_x(3,0)(x-3) + f_y(3,0)(y-0)$$

$$L(x,y) = 2 + \frac{1}{4}(x-3) + y = \frac{1}{4}x + y + \frac{5}{4}$$

DIFFERENTIALS

RECALL: $dy = f'(x)\,dx$



IN 2 VAR:

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\left(\Delta z = z - z_0 \approx f_x(a,b)(x-a) + f_y(a,b)(y-b) \right)$$

$$z \approx z_0 + \ldots$$

THIS IS LINEAR APPROXIMATION!

IT'S JUST THAT WE ESTIMATE Δz, NOT z ITSELF.

ex. IF $z = x^2 - xy + 3y^2$, FIND dz.

IF (x, y) CHANGES FROM $(3,-1)$ TO $(2.96, -0.95)$

COMPARE THE VALUES OF dz & Δz.

ANS $dz = (2x+y)dx + (-x+6y)dy$

$dz = (5)(-.04) + (-9)(0.05) = -.65$

$\Delta z = (3^2 + 3 + 3) - \left((2.96)^2 - (2.96)(-.95) + 3(-.95)^2\right)$

$= 15 - (8.7616 + 2.812 + 2.7075)$

$= 0.7189$

ex. USE DIFFERENTIALS TO ESTIMATE AMOUNT OF METAL IN A CLOSED CYLINDRICAL CAN THAT IS 10 cm HIGH, 4 cm IN DIAMETER IF METAL IN TOP & BOTTOM IS .1 cm THICK & IN SIDE IS 0.05 cm THICK.

TOPS: $V(r,h) = \pi r^2 h$

$V_r = 2\pi r h$, $V_h = \pi r^2$

$V_r(2,10) = 40\pi$, $V_h(2,10) = 4\pi$

$dV = 40\pi(.05) + 4\pi(.2)$

$= 2\pi + .8\pi = \boxed{2.8\pi \text{ cm}^3}$