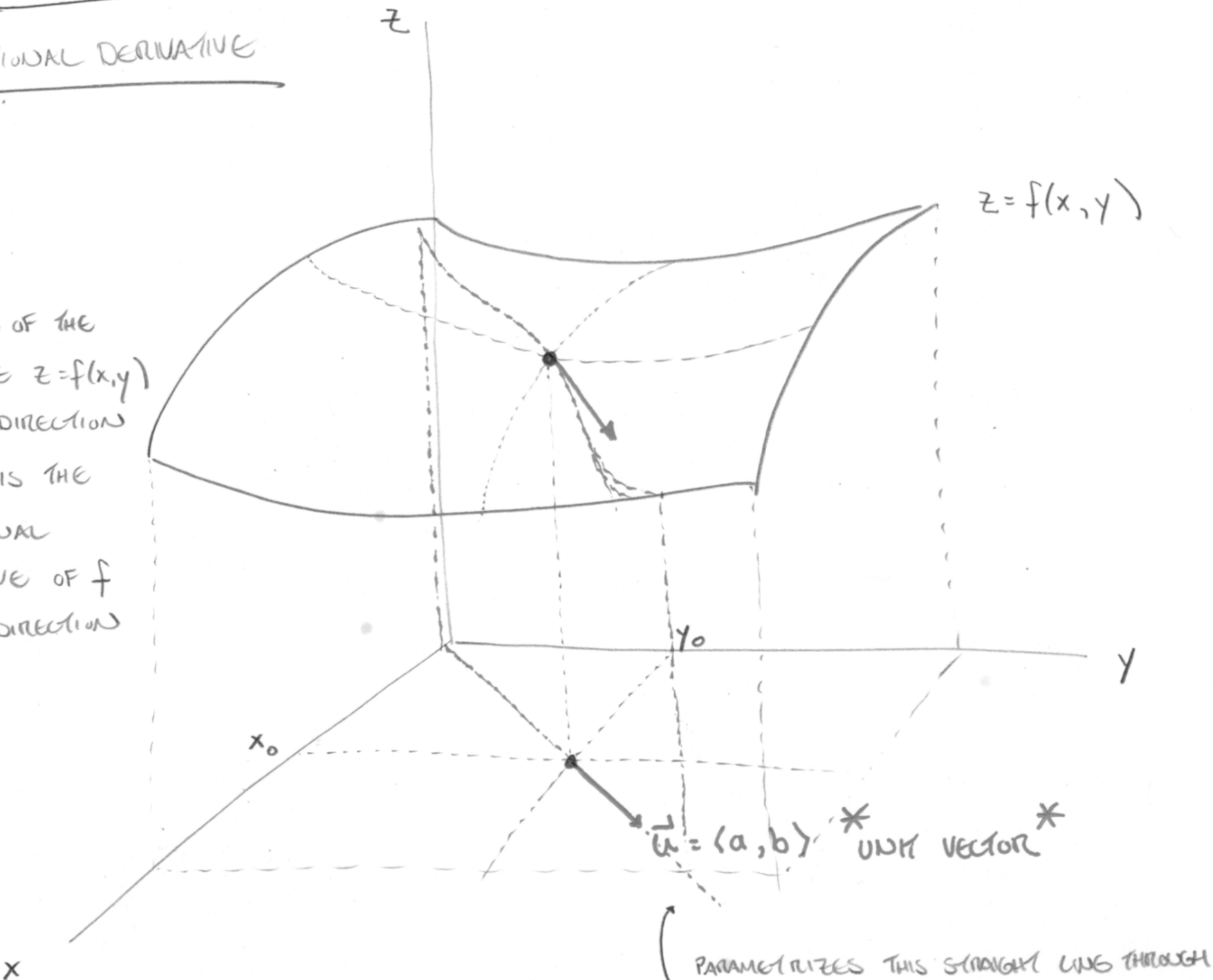
DIRECTIONAL DERIVATIVE

THE SLOPE OF THE SURFACE $z = f(x,y)$ IN THE DIRECTION OF $\vec{u}$ IS THE DIRECTIONAL DERIVATIVE OF f IN THE DIRECTION OF $\vec{u}$.



PARAMETRIZES THIS STRAIGHT LINE THROUGH (x_0, y_0) IN DIRECTION $\vec{u}$.

CONSIDER $z = f(x,y)$, $\underbrace{x = x_0 + at \quad , \quad y = y_0 + bt}$

THEN THE DIRECTIONAL DERIVATIVE AT (x_0, y_0) IN DIRECTION $\vec{u}$ $= \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dy}$

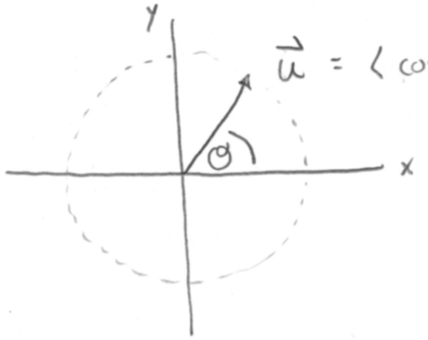
$$\boxed{D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b} \qquad (1)$$

* NOTE THAT $\vec{u} = \langle a, b \rangle$ IS A UNIT VECTOR !

WHY? THE SPEED OF THE PARTICLE PARAMETRIZED BY $x = x_0 + at$, $y = y_0 + bt$ EFFECTS $\dfrac{dz}{dt}$.

BY CONVENTION, WE FORCE THE PARTICLE TO MOVE AT UNIT SPEED.

UNIT VECTORS IN POLAR COORDINATES



$\vec{u} = \langle \cos\theta , \sin\theta \rangle$

$$\boxed{\begin{array}{l}\text{IF } \vec{u} \text{ MAKES ANGLE } \theta \text{ w/ POS. X-AXIS} \\ D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0,y_0)\sin\theta\end{array}} \quad (2)$$

NOTE :

·) WHEN $\theta = 0$, $\vec{u} = \hat{\imath}$

& $D_{\hat{\imath}} f = f_x$

·) WHEN $\theta = \frac{\pi}{2}$, $\vec{u} = \hat{\jmath}$

& $D_{\hat{\jmath}} f = f_y$

e.g. LET $f(x,y) = x^2 + 2xy + 3y^2$, $\vec{u} = \langle \cos\frac{2\pi}{3} , \sin\frac{2\pi}{3} \rangle$

FIND $D_{\vec{u}} f(3,-4)$.

GRADIENT

OBSERVE THAT (1) & (2) CAN BE WRITTEN USING THE DOT PRODUCT.

$$D_{\vec{u}} f(x_0, y_0) = \langle f_x(x_0,y_0) , f_y(x_0,y_0) \rangle \cdot \vec{u}$$

ARBITRARY

OR SIMPLY $D_{\vec{u}} f = \underbrace{\langle f_x , f_y \rangle}_{\text{VECTOR-VALUED FUNCTION CALLED THE GRADIENT OF } f.} \cdot \vec{u}$

Def: IF f IS A FUNCTION OF 2 VARIABLES x, y WITH PARTIAL DERIVATIVES f_x, f_y THEN THE GRADIENT OF f IS THE VECTOR-VALUED FUNCTION

$$\nabla f = \langle f_x , f_y \rangle = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}$$

$$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

e.g. LET $f(x,y) = x^3 - 3x^2 y + 3xy^2 - y^3$

(a) FIND ∇f.

(b) FIND DIRECTIONAL DERIVATIVE OF f AT $(1,2)$ IN DIRECTION $\vec{v} = \langle 5, -12 \rangle$

HIGHER DIMENSIONS:

GIVEN $f(x,y,z)$, $\vec{u} = \langle a, b, c \rangle$, WE HAVE

$$\nabla f = \langle f_x , f_y , f_z \rangle$$

- $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

GIVEN $f(x_1, x_2, \ldots, x_n)$, $\vec{u} = \langle a_1, a_2, \ldots, a_n \rangle$, WE HAVE

$$\nabla f = \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle$$

- $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

MAXIMIZING THE DIRECTIONAL DERIVATIVE

THM. SUPPOSE f IS DIFFERENTIABLE FUNCTION OF 2 OR 3 VARIABLES. THE MAXIMUM VALUE OF THE DIRECTIONAL DERIVATIVE $D_{\vec{u}} f(\vec{x})$, WHERE $\vec{x} = \langle x_0, y_0 \rangle$ OR $\vec{x} = \langle x_0, y_0, z_0 \rangle$, IS $|\nabla f(\vec{x})|$ & IT OCCURS WHEN $\vec{u}$ HAS THE SAME DIRECTION AS $\nabla f(\vec{x})$.

"THE GRADIENT POINTS UP-HILL."

PROOF: WE HAVE

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos\theta$$
$$= |\nabla f| \cos\theta$$

MAXIMIZED WHEN $\cos\theta = 1$, i.e. $\theta = 0$, i.e. $\vec{u} \parallel \nabla f$. $\square$

e.g. SUPPOSE THAT OVER A CERTAIN REGION OF SPACE, THE ELECTRICAL POTENTIAL V IS GIVEN BY $V(x,y,z) = 5x^2 - 3xy + xyz$.

$\frac{32}{\sqrt{3}}$	(a) FIND THE RATE OF CHANGE OF POTENTIAL AT $P(3,4,5)$ IN THE DIRECTION $\vec{v} = \hat{i} + \hat{j} - \hat{k}$
$\langle 38, 6, 12 \rangle$	(b) IN WHICH DIRECTION DOES V CHANGE MOST RAPIDLY?
$2\sqrt{406}$	(c) WHAT IS THE MAXIMUM RATE OF CHANGE AT P?

TANGENT PLANES TO LEVEL SURFACES

CONSIDER A CURVE $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

ON A LEVEL SURFACE $F(x, y, z) = k$ $\quad$ (e.g. $x^2 + y^2 + z^2 = 1$)

THAT PASSES THROUGH $P(x_0, y_0, z_0) = (x(t_0), y(t_0), z(t_0))$.

SINCE $\vec{r}(t)$ LIES ON THE LEVEL SURFACE, WE HAVE

$$F(x(t), y(t), z(t)) = k$$

$$\Big\downarrow \frac{d}{dt}$$

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt} = 0$$

i.e. $\quad \nabla F \cdot \vec{r}\,'(t) = 0$.

IN PARTICULAR, WHEN $t = t_0$ WE HAVE

$$\underbrace{\nabla F(x_0, y_0, z_0)} \cdot \underbrace{\vec{r}\,'(t_0)} = 0$$

THE GRADIENT AT P IS $\perp$ TO THE TANGENT VECTOR OF ANY CURVE ON THE SURFACE THAT PASSES THROUGH P.

$\therefore$ IF $\nabla F(x_0, y_0, z_0) \neq \vec{0}$, THEN $\nabla F(x_0, y_0, z_0)$ GIVES A NORMAL VECTOR TO THE TANGENT PLANE TO THE LEVEL SURFACE $F(x, y, z) = k$ AT $P(x_0, y_0, z_0)$.