§11.6 Directional Derivatives & the Gradient Vector

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1. $f(x,y) = ye^{-x}$, $(0,4)$, $\theta = \frac{2\pi}{3}$

$$\nabla f\Big|_{(0,4)} \cdot \left\langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3} \right\rangle = \langle -ye^{-x}, e^{-x} \rangle\Big|_{(0,4)} \cdot \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$= \langle -4, 1 \rangle \cdot \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = 2 + \frac{\sqrt{3}}{2} = \boxed{\frac{4+\sqrt{3}}{2}}$$

3. $f(x,y) = \sin(2x+3y)$, $P(-6,4)$, $\vec{u} = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

(a) $\boxed{\nabla f = \langle 2\cos(2x+3y), 3\cos(2x+3y) \rangle}$

(b) $\boxed{\langle 2, 3 \rangle}$

(c) $\langle 2, 3 \rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle = \sqrt{3} - \frac{3}{2} = \boxed{\frac{2\sqrt{3}-3}{2}}$

5. $f(x,y,z) = x^2yz - xyz^3$, $P(2,-1,1)$, $\vec{u} = \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle$

(a) $\boxed{\nabla f = \langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2 \rangle}$

(b) $\langle -4+1, 4-2, -4+6 \rangle = \boxed{\langle -3, 2, 2 \rangle}$

(c) $\langle -3, 2, 2 \rangle \cdot \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle = 0 + \frac{8}{5} - \frac{6}{5} = \boxed{\frac{2}{5}}$

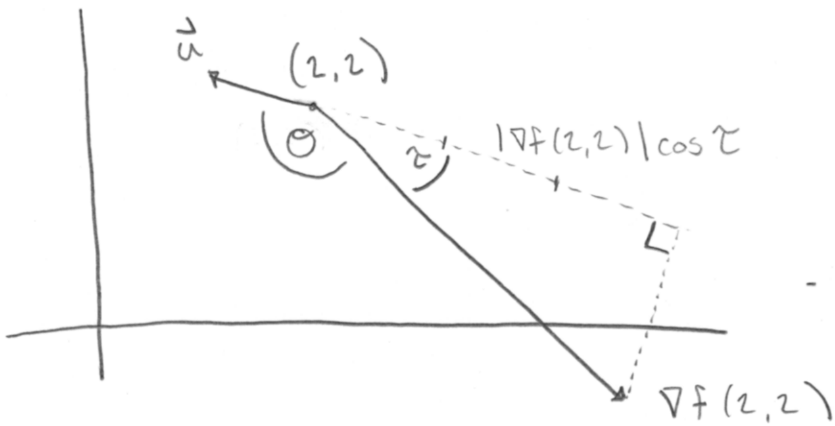
8. $\nabla f = \left\langle \dfrac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} , \dfrac{-2xy}{(x^2+y^2)^2} \right\rangle$

$\nabla f|_{(1,2)} = \left\langle \dfrac{3}{25} , \dfrac{-4}{25} \right\rangle$

$\vec{v} = \langle 3,5 \rangle \Rightarrow \vec{u} = \dfrac{1}{|\vec{v}|}\vec{v} = \dfrac{1}{\sqrt{34}} \langle 3,5 \rangle$

$\left\langle \dfrac{3}{25} , \dfrac{-4}{25} \right\rangle \cdot \dfrac{1}{\sqrt{34}} \langle 3,5 \rangle = \boxed{\dfrac{1}{\sqrt{34}} \left\langle \dfrac{9}{25} , \dfrac{-20}{25} \right\rangle}$

12.



$\nabla f(2,2) \cdot \vec{u} = |\nabla f(2,2)| \underbrace{|\vec{u}|}_{1} \cos \theta$

$= |\nabla f(2,2) \cos \theta = -|\nabla f(2,2)| \cos \tau$

$\boxed{\approx -3}$

14. $\nabla f = \langle y+z,\ x+z,\ x+y \rangle$

$\nabla f(1,-1,3) = \langle 2, 4, 0 \rangle$

$\vec{v} = \overrightarrow{PQ} = \vec{Q} - \vec{P} = \langle 2-1,\ 4+1,\ 5-3 \rangle = \langle 1, 5, 2 \rangle$

$\vec{u} = \dfrac{1}{|\vec{v}|}\vec{v} = \dfrac{1}{\sqrt{30}}\langle 1, 5, 2 \rangle$

$\nabla f(2,4,0) \cdot \vec{u} = \dfrac{2+20}{\sqrt{30}} = \boxed{\dfrac{22}{\sqrt{30}}}$

16. $\nabla f(0,2) = \left\langle t^2 e^{st},\ (1+st)e^{st} \right\rangle \Big|_{(0,2)} = \langle 4, 1 \rangle$

MAX RATE OF CHANGE IN f AT $(0,2) = |\nabla f(0,2)| = \sqrt{17}$

IN DIRECTION $\nabla f(0,2) = \langle 4, 1 \rangle$

18. $\nabla f(1,2,1) = \left\langle \dfrac{qr}{1+(pqr)^2},\ \dfrac{pr}{1+(pqr)^2},\ \dfrac{qr}{1+(pqr)^2} \right\rangle \Big|_{(1,2,1)}$

$= \boxed{\left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \right\rangle}$ ← DIRECTION OF MAX RATE OF CHANGE

$\left|\left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \right\rangle\right| = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \boxed{\dfrac{3}{5}}$ ← MAX RATE OF CHANGE

19. MINIMIZE $\nabla f(\vec{x}) \cdot \vec{u} = |\nabla f(\vec{x})|\underbrace{|\vec{u}|}_{1} \cos\theta$

MINIMUM WHEN $\theta = \pi$

i.e. WHEN $\nabla f(\vec{x})$ & $\vec{u}$ POINT IN OPPOSITE DIRECTIONS

$(\cos\pi = -1)$

(b) $-\nabla f(2,-3) = -\langle 4x^3y - 2xy^3, x^4 - 3x^2y^2 \rangle \Big|_{(2,-3)}$

$= -\langle -96 + 108, 16 - 108 \rangle = \boxed{\langle -12, 92 \rangle}$

20. $\nabla f(1,0) = \langle 2x + y\cos xy, x\cos xy \rangle \Big|_{(1,0)} = \langle 2, 1 \rangle$

SOLVE $\nabla f(1,0) \cdot \vec{u} = 1$ LET $\vec{u} = \langle a, b \rangle$

$\langle 2,1 \rangle \cdot \langle a, b \rangle = 1$

$2a + b = 1$

$a^2 + b^2 = 1$

$b = \pm\sqrt{1-a^2}$

$2a \pm \sqrt{1-a^2} = 1$

$\pm\sqrt{1-a^2} = 1 - 2a$ SQUARE BOTH SIDES

$1 - a^2 = 1 - 4a + 4a^2$

$0 = a(-4 + 5a)$

$a = 0$
$b = 1$

$a = \frac{4}{5}$
$b = \frac{3}{5}$

$\boxed{\langle 0, 1 \rangle \text{ \& } \langle \tfrac{4}{5}, \tfrac{3}{5} \rangle}$

21. $\nabla f = \langle 2x-2,\ 2y-4 \rangle = k\langle 1, 1 \rangle$

$$\Rightarrow \quad 2x-2 = 2y-4$$

$$2x+2 = 2y$$

$$\boxed{y = x + 1}$$

SOLUTIONS ARE POINTS ON THIS LINE.

26. $f(x,y) = 1000 - .005x^2 - .01y^2$

$\nabla f(x,y) = \langle -0.01x,\ -0.02y \rangle \rightarrow \nabla f(60,40) = \langle -.6,\ .8 \rangle$

(a) $\langle -.6, .8 \rangle \cdot \langle 0, -1 \rangle = -.8$ NEGATIVE $\rightarrow$ $\boxed{\text{DESCEND AT RATE } .8\ m/m}$

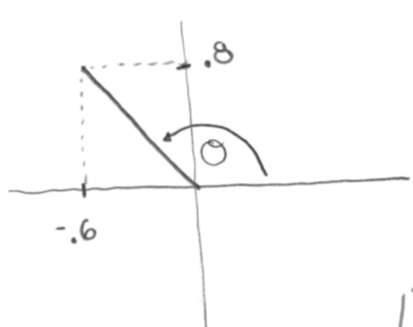
UNIT VECTOR NORTHWEST

(b) $\langle -.6, .8 \rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = -0.3\sqrt{2} + 0.4\sqrt{2} = 0.1\sqrt{2}$ POSITIVE

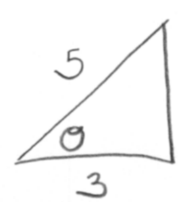
$\boxed{\text{ASCEND AT RATE } 0.1\sqrt{2}\ m/m}$

(c) SLOPE LARGEST IN DIRECTION $\boxed{\langle -.6, .8 \rangle}$

RATE OF ASCENT IN THIS DIRECTION IS $|\langle -.6, .8 \rangle| = \boxed{1\ m/m}$



$\tan\theta = \frac{.8}{-.6} = -\frac{4}{3}$



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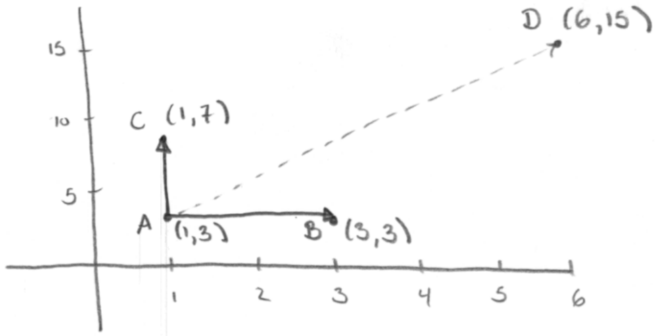
$$\boxed{\theta = \tan^{-1}\left(-\frac{4}{3}\right)}$$

27.



$$\vec{AD} = \langle 5, 12 \rangle \longrightarrow \vec{u} = \frac{1}{|\langle 5,12\rangle|}\langle 5,12\rangle = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$$

$$D_u f(1,3) = \nabla f(1,3) \cdot \left(\frac{5}{13}\hat{i} + \frac{12}{13}\hat{j}\right)$$

$$= \frac{5}{13} D_{\hat{i}} f(1,3) + \frac{12}{13} D_{\hat{j}} f(1,3)$$

$$= \frac{5}{13}(3) + \frac{12}{13}(26) = \frac{15}{13} + 24 = \boxed{\frac{327}{13}}$$

29. (a) $\nabla(au+bv) = \left\langle \frac{\partial}{\partial x}(au+bv), \frac{\partial}{\partial y}(au+bv) \right\rangle$

$$= \left\langle \frac{\partial}{\partial x} au, \frac{\partial}{\partial y} au \right\rangle + \left\langle \frac{\partial}{\partial x} bv, \frac{\partial}{\partial y} bv \right\rangle$$

$$= a\left\langle \frac{\partial}{\partial x} u, \frac{\partial}{\partial y} u \right\rangle + b\left\langle \frac{\partial}{\partial x} v, \frac{\partial}{\partial y} v \right\rangle = a\nabla u + b\nabla v \quad \checkmark$$

(b) $\nabla uv = \left\langle \frac{\partial}{\partial x} uv, \frac{\partial}{\partial y} uv \right\rangle = \left\langle \frac{\partial}{\partial x}[u]v + u\frac{\partial}{\partial x}[v], \frac{\partial}{\partial y}[u]v + u\frac{\partial}{\partial y}[v] \right\rangle$

$$= \left\langle \frac{\partial}{\partial x}[u]v, \frac{\partial}{\partial y}[u]v \right\rangle + \left\langle u\frac{\partial}{\partial x}[v], u\frac{\partial}{\partial y}[v] \right\rangle$$

$$= \left\langle \frac{\partial}{\partial x} u, \frac{\partial}{\partial y} u \right\rangle v + u\left\langle \frac{\partial}{\partial x} v, \frac{\partial}{\partial y} v \right\rangle = (\nabla u)v + u(\nabla v) \quad \checkmark$$

(c) $$\nabla\left(\frac{u}{v}\right) = \left\langle \frac{\partial}{\partial x}\left(\frac{u}{v}\right), \frac{\partial}{\partial y}\left(\frac{u}{v}\right)\right\rangle$$

$$= \left\langle \frac{v\frac{\partial}{\partial x}u - u\frac{\partial}{\partial x}v}{v^2}, \frac{v\frac{\partial}{\partial y}u - u\frac{\partial}{\partial y}v}{v^2} \right\rangle$$

$$= \left\langle \frac{v\frac{\partial}{\partial x}u}{v^2}, \frac{v\frac{\partial}{\partial y}u}{v^2}\right\rangle - \left\langle \frac{u\frac{\partial}{\partial x}v}{v^2}, \frac{u\frac{\partial}{\partial y}v}{v^2}\right\rangle$$

$$= \frac{v}{v^2}\left\langle \frac{\partial}{\partial x}u, \frac{\partial}{\partial y}u\right\rangle - \frac{u}{v}\left\langle \frac{\partial}{\partial x}v, \frac{\partial}{\partial y}v\right\rangle = \frac{v\nabla u - u\nabla v}{v^2}$$ ✓

(d) $$\nabla u^n = \left\langle \frac{\partial}{\partial x}u^n, \frac{\partial}{\partial y}u^n\right\rangle = \left\langle nu^{n-1}\frac{\partial}{\partial x}u, nu^{n-1}\frac{\partial}{\partial y}u\right\rangle$$

$$= nu^{n-1}\left\langle \frac{\partial}{\partial x}u, \frac{\partial}{\partial y}u\right\rangle = nu^{n-1}\nabla u$$ ✓

32. $$F(x,y,z) = x^2 - y - z^2 = 0$$

$\vec{n}$

$$\nabla F(4,7,3) = \langle 2, -1, -2z\rangle\Big|_{(4,7,3)} = \langle 2, -1, -6\rangle$$

$$\boxed{\begin{array}{c} 2(x-4) - (y-7) - 6(z-3) = 0 \\ \vec{n} = \langle 2, -1, -6\rangle \end{array}}$$

GRADIENTS POINT $\perp$ TO LEVEL SURFACES.

36. $F(x,y,z) = x^4 + y^4 + z^4 - 3x^2y^2z^2 = 0$

$$\nabla F(1,1,1) = \langle 4x^3 - 6xy^2z^2,\ 4y^3 - 6x^2yz^2,\ 4z^3 - 6x^2y^2z \rangle \Big|_{(1,1,1)}$$

$$= \langle -2, -2, -2 \rangle$$

$$\boxed{\begin{array}{c} -2(x-1) - 2(y-1) - 2(z-1) = 0 \\ \vec{n} = \langle -2, -2, -2 \rangle \end{array}}$$

39. GRADIENTS POINT $\perp$ TO LEVEL CURVES.

$\nabla f(3,2) = \langle y, x \rangle \Big|_{(3,2)} = \langle 2, 3 \rangle$ ← IS $\perp$ TO TANGENT LINE

$\langle 2, 3 \rangle$ HAS SLOPE $\frac{3}{2}$

$\Rightarrow$ SLOPE OF TANGENT LINE IS $-\frac{2}{3}$.

$\therefore \quad y - 2 = -\frac{2}{3}(x - 3)$ POINT-SLOPE FORMULA.

OR $\boxed{y = -\frac{2}{3}x + 4}$

42. $F(x,y,z) = x^2 - y + z^2 = 0$

$\nabla F = \langle 2x, -1, 2z \rangle$

↑ WHEN IS THIS $\parallel$ TO $\langle 1, 2, 3 \rangle$

SOLVE: $\langle 2x, -1, 2z \rangle = k \langle 1, 2, 3 \rangle$ NOTE THAT $k = -\frac{1}{2}$

$2x = -1 \qquad 2z = -\frac{3}{2}$

$x = -\frac{1}{2} \qquad z = -\frac{3}{4}$

$\therefore \boxed{\left(-\frac{1}{2}, \frac{13}{16}, -\frac{3}{4}\right)}$

$y = \left(-\frac{1}{2}\right)^2 + \left(-\frac{3}{4}\right)^2 = \frac{1}{4} + \frac{9}{16}$