11.7 MAXIMUM & MINIMUM VALUES

DEF: $f(x,y)$ HAS LOCAL MAX AT (a,b) IF $f(a,b) \geq f(x,y)$ FOR ALL (x,y) NEAR (a,b). $f(a,b)$ IS CALLED LOCAL MAX VALUE. SIMILARLY, LOCAL MIN, LOCAL MIN VALUE.

DEF: MAX/MIN IS ABSOLUTE IF INEQUALITY HOLDS OVER ENTIRE DOMAIN.

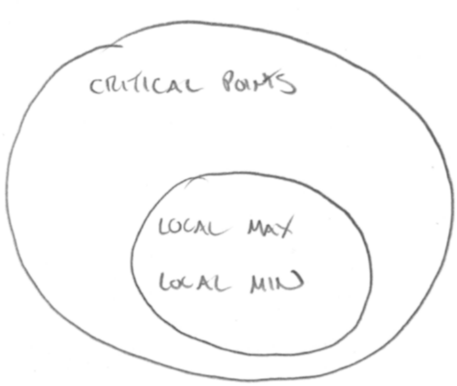
THM: IF f HAS A LOCAL MAX/MIN AT (a,b) & $f_x(a,b)$, $f_y(a,b)$ EXIST, THEN $f_x(a,b) = f_y(a,b) = 0$

Pf: INTUITION. e.g. "SUPPOSE NOT..."

"TANGENT PLANE AT LOCAL MAX/MIN IS HORIZONTAL"

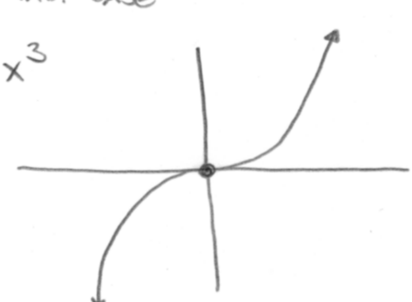
DEF: A CRITICAL POINT OF $f(x,y)$ IS A POINT (a,b) S.T.

- $f_x(a,b) = f_y(a,b) = 0$, OR
- AT LEAST ONE PARTIAL DERIV. D.N.E. AT (a,b).



RECALL 1-VAR. CASE

$f(x) = x^3$



e.g. $f(x,y) = -2 + (2x-1)^2 + (3y-2)^2$ PARABOLOID!

$= -2 + 4x^2 - 4x + 1 + 9y^2 - 12y + 4$

$$\boxed{f(x,y) = 4x^2 - 4x + 9y^2 - 12y + 3}$$

$f_x = 8x - 4 = 0$ $\quad$ $f_y = 18y - 12 = 0$

$\underline{x = \frac{1}{2}}$ $\quad$ $\underline{y = \frac{2}{3}}$ ← CRITICAL POINT

→ LOCAL MIN BY GEOMETRY

e.g. $f(x,y) = x^2 - y^2$ → CRIT. PNT. $(0,0)$ → NEITHER MAX/MIN BY GEOMETRY. (PLANES $x=0$, $y=0$)

MEMORIZE THESE PICTURES!!

"SADDLE POINT"

SECOND DERIVATIVE TEST

SUPPOSE SECOND PARTIAL DERIVATIVES OF f ARE CONTINUOUS ON A DISK WITH CENTER (a,b) & SUPPOSE $f_x(a,b) = 0$ & $f_y(a,b) = 0$ (CRIT. PNT.)

DEFINE

$$D = D(a,b) := f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

- IF $D > 0$ AND $f_{xx}(a,b) > 0$ THEN LOCAL MIN
- IF $D > 0$ AND $f_{xx}(a,b) < 0$ THEN LOCAL MAX
- IF $D < 0$ THEN $f(a,b)$ IS NOT LOCAL EXTREME VAL.

PROOF OF PART (a)

WE WANT TO SHOW THAT $D_{\vec{u}}^2 f(a,b) > 0$ FOR ALL UNIT VECTORS $\vec{u}$.

THAT IS, GIVEN ANY PLANAR CURVE ON $z = f(x,y)$ THAT PASSES THROUGH $(a, b, f(a,b))$, THE CURVE IS CONCAVE UP.

LET $\vec{u} = \langle h, k \rangle$ & RECALL $D_{\vec{u}} f = \frac{\partial}{\partial x} f \cdot h + \frac{\partial}{\partial y} f \cdot k$

$$D_{\vec{u}}^2 f = D_{\vec{u}}(D_{\vec{u}} f) = \frac{\partial}{\partial x}(D_{\vec{u}} f) \cdot h + \frac{\partial}{\partial y}(D_{\vec{u}} f) \cdot k$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f \cdot h + \frac{\partial}{\partial y} f \cdot k \right) \cdot h + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f \cdot h + \frac{\partial}{\partial y} f \cdot k \right) \cdot k$$

$$= \frac{\partial^2 f}{\partial x^2} \cdot h^2 + \frac{\partial^2 f}{\partial x \partial y} \cdot hk + \frac{\partial^2 f}{\partial y \partial x} \cdot hk + \frac{\partial^2 f}{\partial y^2} \cdot k^2$$

$$= f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2$$

[CLAIRAUT'S THM: IF f_{xy} & f_{yx} ARE CONTINUOUS ON A NBHD OF (a,b)

THEN $f_{xy}(a,b) = f_{yx}(a,b)$]

$$= f_{xx} \left[h^2 + 2 \cdot \frac{f_{xy}}{f_{xx}} hk \right] + f_{yy} k^2$$

$$= f_{xx} \left(h + \frac{f_{xy}}{f_{xx}} k \right)^2 + f_{yy} k^2 - \frac{f_{xy}^2}{f_{xx}} k^2$$

$$= \underbrace{f_{xx}}_{\text{POS.}} \underbrace{\left(h + \frac{f_{xy}}{f_{xx}} k \right)^2}_{\text{NON-NEG. } *} + \underbrace{\frac{k^2}{f_{xx}}}_{\text{NON-NEG } *} \underbrace{\left(f_{xx} f_{yy} - f_{xy}^2 \right)}_{\text{POS.}} > 0$$

SINCE * CAN'T BOTH BE 0.

11.7.2

e.g. FIND THE LOCAL MAX'S, MIN'S, & SADDLE POINTS OF

$$f(x,y) = y^3 + 3x^2 y - 6x^2 - 6y^2 + 2$$

e.g. FIND POINT(S) ON CONE $z^2 = x^2 + y^2$ CLOSEST TO THE POINT $(4, 2, 0)$.

e.g. FIND DIMENSIONS OF BOX THAT MAXIMIZES VOLUME IF TOTAL SURFACE AREA $= 64 \text{ cm}^3$.

ABS. MAX/MIN VALUES

(RECALL EXTREME VAL. THM IN 1 VAR.)

EXTREME VALUE THM:

IF f IS CONTINUOUS ON $\overbrace{\text{CLOSED, BOUNDED SET } D \text{ IN } \mathbb{R}^n}^{\text{"COMPACT"}}$

THEN f ATTAINS AN ABSOLUTE MAX VALUE $f(x_1, y_1)$

& ABSOLUTE MIN VALUE $f(x_2, y_2)$ AT SOME POINTS

(x_1, y_2) & (x_2, y_2) IN D.

CLOSED SETS CONTAIN THEIR BOUNDARY, BOUNDED SETS FIT INSIDE A DISK CENTERED AT O.

$\{(x,y) : 0 \le x \le 3, -1 \le y \le 1\}$

e.g.

$\{(x,y) : (x-3)^2 + (y+1)^2 \le 2\}$ $\{(x,y) : \frac{1}{3}(x-2)^2 + \frac{1}{4}(x+1)^2 \le 1\}$

CLOSED & BOUNDED

e.g.

NOT CLOSED

NOT BOUNDED.

TO FIND ABSOLUTE MAX/MIN VALUES OF

CONTINUOUS f ON CLOSED & BOUNDED D:

1. FIND CRITICAL POINTS & VALUES OF f

   AT CRITICAL POINTS (CRITICAL VALUES)

2. FIND EXTREME VALUES ON BOUNDARY (1 VARIABLE)

3. ABS MAX = MAX {VALUES IN 1 & 2}

   ABS MIN = MIN {VALUES IN 1 & 2}

e.g. FIND ABS. MAX/MIN VALUES OF $f(x,y) = 4x + 6y - x^2 - y^2$

ON $D = \{(x,y) : 0 \le x \le 4,\ 0 \le y \le 5\}$