§11.7 MAXIMUMS & MINIMUMS

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3. $f(x,y) = x^2 + xy + y^2 + y$

① CRIT. PNTS:

$$f_x = 2x + y = 0 \qquad\qquad f_y = x + 2y + 1 = 0$$

$$y = -2x \longrightarrow x + 2(-2x) + 1 = 0$$

$$-3x = -1$$

$$x = \frac{1}{3},\ y = -\frac{2}{3}$$

ONLY ONE CRIT. PNT: $\underline{\left(\frac{1}{3}, -\frac{2}{3}\right)}$

② 2nd DERIV'S TEST:

$$f_{xx} = 2 \qquad f_{yy} = 2 \qquad f_{xy} = 1$$

$$\left.\begin{array}{l} D = (2)(2) - (1)^2 = 3 > 0 \\ f_{xx} > 0 \end{array}\right\} \Rightarrow \boxed{\begin{array}{c} \left(\frac{1}{3}, -\frac{2}{3}\right) \text{ IS LOCAL MIN} \\ \text{W/ MIN VALUE } f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3} \end{array}}$$

• 7. $f(x,y) = x^3 - 12xy + 8y^3$

$$f_x = 3x^2 - 12y = 0 \qquad\qquad f_y = -12x + 24y^2 = 0$$

$$x^2 = 4y \longleftarrow 2y^2 = x$$

$$(2y^2)^2 = 4y$$

$$y^4 - y = 0$$

$$y(y^3 - 1) = 0$$

$\underline{y = 0}$ OR $\underline{y = 1}$

$$y = 0 \Rightarrow x = 0 \rightarrow (0,0)$$

$$y = 1 \Rightarrow x = 2 \rightarrow (2,1)$$

$f_{xx} = 6x \qquad f_{yy} = 48y \qquad f_{xy} = -12$

$(0,0)$: $D = (0)(0) - (-12)^2 < 0 \longrightarrow \boxed{\text{SADDLE PNT AT } (0,0)}$

$(2,1)$: $D = (12)(48) - (-12)^2 > 0$, $f_{xx}(2,1) = 12 > 0$

$$\longrightarrow \boxed{\begin{array}{l}\text{LOCAL MIN AT } (2,1) \text{ w/}\\ \text{LOCAL MIN VALUE } f(2,1) = 8 - 24 + 8 = -8\end{array}}$$

11. $f(x,y) = (x^2+y^2)e^{y^2-x^2}$

$$f_x = 2x e^{y^2-x^2} - 2x(x^2+y^2)e^{y^2-x^2}$$

$$= 2x(1-x^2-\underset{\uparrow\, =0}{y^2})\underset{\uparrow\ \text{NEVER } 0}{e^{y^2-x^2}} = 0$$

$$\Rightarrow 2x(1-x^2) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = \pm 1$$

$$f_y = 2y e^{y^2-x^2} + 2y(x^2+y^2)e^{y^2-x^2}$$

$$= 2y\underset{\nwarrow}{(1+x^2+y^2)}\underset{\uparrow}{e^{y^2-x^2}} = 0$$

$$\Rightarrow y = 0 \qquad \text{NEVER } 0$$

$\therefore$ CRIT. PNTS: $\underline{\underline{(0,0),\ (1,0),\ (-1,0)}}$

$$f_{xx} = 2\left[(1-x^2-y^2)e^{y^2-x^2} - 2x^2 e^{y^2-x^2} - 2x^2(1-x^2-y^2)e^{y^2-x^2}\right]$$

$$= 2(\overset{\checkmark}{1} - \overset{\checkmark}{x^2} - \overset{\checkmark}{y^2} - \overset{\checkmark}{2x^2} - \overset{\checkmark}{2x^2} + 2x^4 + 2x^2y^2)e^{y^2-x^2}$$

$$= 2(1-5x^2-y^2+2x^4+2x^2y^2)e^{y^2-x^2}$$

$$f_{yy} = 2\left[(1+x^2+y^2) + 2y^2 + 2y^2(1+x^2+y^2)\right]e^{y^2-x^2}$$

$$= 2(\overset{\checkmark}{1} + \overset{\checkmark}{x^2} + \overset{\checkmark}{y^2} + \overset{\checkmark}{2y^2} + \overset{\checkmark}{2y^2} + 2x^2y^2 + 2y^4)e^{y^2-x^2}$$

$$= 2(1+x^2+5y^2+2y^4+2x^2y^2)e^{y^2-x^2}$$

$$f_{xy} = \cancel{4xye^{y^2-x^2}} - \cancel{4xye^{y^2-x^2}} - 4xy(x^2+y^2)e^{y^2-x^2}$$

$(0,0)$: $D = (2)(2) - (0)^2 = 4 > 0$

$f_{xx}(0,0) = 2 > 0 \longrightarrow \boxed{\text{(0,0) IS LOCAL MIN W/ LOCAL MIN VAL } f(0,0) = 0}$

$(1,0)$ & $(-1,0)$: $D = (-4e^{-1})(4e^{-1}) - (\quad)^2 < 0 \longrightarrow \boxed{\text{(1,0) \& (-1,0) ARE SADDLE POINTS.}}$

13. $f(x,y) = y^2 - 2y\cos x \quad , \quad -1 \le x \le 7$

$f_x = 2y\sin x = 0$

$f_y = 2y - 2\cos x = 0$

$y = \cos x$

$y = 0$ or $x = 0, \pi, 2\pi$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad y = 1 \quad y = -1$

CRITICAL PNTS: $(0,1), (\frac{\pi}{2}, 0), (\frac{3\pi}{2}, 0), (\pi, -1), (2\pi, 1)$

$f_{xx} = 2y\cos x \qquad f_{yy} = 2 \qquad f_{xy} = 2\sin x$

$(0,1)$: $D = (2)(2) - 0 = 4 > 0$, $f_{xx} > 0 \longrightarrow \boxed{\text{LOCAL MIN AT (0,1) W/ LOCAL MIN VAL } f(0,1) = -1}$

$(\frac{\pi}{2}, 0)$: $D = (0)(2) - (2)^2 < 0 \longrightarrow \boxed{\text{SADDLE PNT AT } (\frac{\pi}{2}, 0)}$

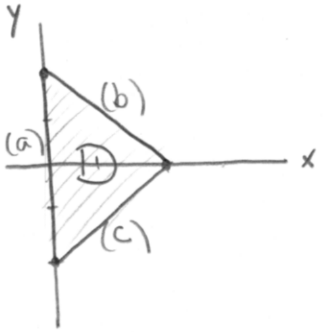
$(\frac{3\pi}{2}, 0)$: $D = (0)(2) - (-2)^2 < 0 \longrightarrow \boxed{\text{SADDLE PNT AT } (\frac{3\pi}{2}, 0)}$

$(\pi, -1)$: $D = (2)(2) - (0)^2 > 0 \longrightarrow f_{xx} > 0 \longrightarrow \boxed{\text{LOCAL MIN AT } (\pi,-1) \text{ W/ LOCAL MIN VAL } f(\pi,-1) = -1}$

$(2\pi, 1)$: $D = (2)(2) - (0)^2 > 0 \longrightarrow f_{xx} > 0 \longrightarrow \boxed{\text{LOCAL MIN AT } (2\pi,1) \text{ W/ LOCAL MIN VAL } f(2\pi,1) = -1}$

23. $f(x,y) = x^2 + y^2 - 2x = (x-1)^2 + y^2 - 1$

HAS LOCAL MIN AT $(1,0)$ w/ MIN VAL $f(1,0) = -1$.



(a) $x=0,\ y \in [-2,2]$

$g(y) = f(0,y) = y^2$

LOCAL MIN @ $(0,0)$ w/ MIN VAL $f(0,0)=0$

ABS MAX @ $(0,\pm 2)$ w/ MAX VAL $f(0,\pm 2) = 4$.

(b) $y = 2 - x\ ,\ x \in [0,2]$

$h(x) = f(x, 2-x) = x^2 + (2-x)^2 - 2x = 2x^2 - 6x + 4$

$h'(x) = 4x - 6 = 0 \longrightarrow x = \frac{3}{2}$ LOCAL MIN $\left(\frac{3}{2}, \frac{1}{2}\right)$

w/ LOCAL MIN VAL $f\left(\frac{3}{2}, \frac{1}{2}\right) = h\left(\frac{3}{2}\right) = -\frac{1}{2}$

$h(0) = 4,\ h(2) = 0 \longrightarrow$ ABS. MAX AT $(0,2)$ w/ VAL $f(0,2) = 4$

(c) $y = x - 2\ ,\ x \in [0,2]$

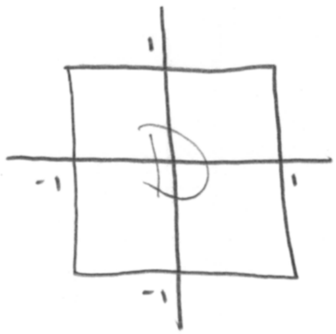
$j(x) = f(x, x-2) = x^2 + (x-2)^2 - 2x =$ SAME AS $h(x)$!

LOCAL MIN AT $\left(\frac{3}{2}, -\frac{1}{2}\right)$ w/ MIN VAL $-\frac{1}{2}$

ABS MAX AT $(0,-2)$ w/ MAX VAL 4.

PNT	VAL	
$(1,0)$	-1	← ABS. MIN VALUE -1 @ $(1,0)$
$(0,0)$	0	
$(0,\pm 2)$	4	← ABS MAX VALUE 4 @ $(0,\pm 2)$
$\left(\frac{3}{2}, \pm\frac{1}{2}\right)$	$-\frac{1}{2}$	

25. $f(x,y) = x^2 + y^2 + x^2y + 4$



$f_x = 2x + 2xy = 0$

$2x(1+y) = 0$

$\swarrow \quad \searrow$

$x = 0 \qquad y = -1$

$f_y = 2y + x^2 = 0$

$x = 0: \quad 2y = 0 \Rightarrow y = 0 \qquad (0,0)$

$y = -1: \quad -2 + x^2 = 0 \rightarrow x = \pm\sqrt{2} \quad (\pm\sqrt{2}, -1)$

$f_{xx} = 2 + 2y \qquad f_{yy} = 2 \qquad f_{xy} = 2x$

$D(0,0) = (2)(2) - (0)^2 > 0 \ , \ f_{xx}(0,0) = 2 > 0 \rightarrow$ LOCAL MIN $(0,0)$

W/ MIN VAL 4

$D(\pm\sqrt{2}, -1) = (0)(2) - 8 < 0 \rightarrow$ SADDLE PNTS @ $(\pm 2, -1)$

ALONG $x = \pm 1$, $y \in [-1,1]$: $f(\pm 1, y) = y^2 + y + 5$

$f'(y) = 2y + 1 = 0 \rightarrow y = -\frac{1}{2} \rightarrow f(\pm 1, -\frac{1}{2}) = 4\frac{3}{4}$ MIN

ALONG $y = 1$, $x \in [-1,1]$: $f(x,1) = 2x^2 + 5$

$f'(x) = 4x = 0 \rightarrow x = 0 \rightarrow f(0,1) = 5$ MIN

ALONG $y = -1$, $x \in [-1,1]$: $f(x,-1) = 5$ CONSTANT

PNTS	VAL
$(0,0)$	4
$(\pm 1, -\frac{1}{2})$	$4\frac{3}{4}$
$(x, -1)$	5
OTHER CORNERS → $(\pm 1, 1)$	7

ABS MIN VAL 4 @ $(0,0)$

ABS MAX VAL 7 @ $(\pm 1, 1)$

31. DISTANCE BETWEEN POINT $\underbrace{\langle x, y, 1-x-y \rangle}_{\text{PNT. ON PLANE}}$ & POINT $\langle 2, 0, -3 \rangle$

$$x + y + z = 1$$

IS GIVEN BY $d(x,y) = \sqrt{(x-2)^2 + (y-0)^2 + (1-x-y+3)^2}$.

$\Rightarrow$ MINIMIZE $d^2(x,y) = (x-2)^2 + y^2 + (4-x-y)^2$

$$= x^2 - 4x + 4 + y^2 + 16 - 4x - 4y - 4x + x^2 + xy - 4y + xy + y^2$$

$$= 2x^2 + 2y^2 + 2xy - 12x - 8y + 20$$

$d_x = 4x + 2y - 12 = 0$ $\qquad$ $d_y = 4y + 2x - 8 = 0$

SYSTEM : $\quad 4x + 2y = 12$

$$\begin{array}{r} -2(\;2x + 4y = 8\;) \\ \hline -6y = -4 \end{array}$$

CRITICAL PNT.

$y = \frac{2}{3} \rightarrow x = \frac{8}{3} \rightarrow \left(\frac{8}{3}, \frac{2}{3}\right)$

$d_{xx} = 4 > 0$ $\qquad$ $d_{yy} = 4$ $\qquad$ $d_{xy} = 2$ $\longrightarrow D\left(\frac{8}{3}, \frac{2}{3}\right) > 0$

$\therefore \left(\frac{8}{3}, \frac{2}{3}\right)$ IS LOCAL MIN

$\therefore$ SHORTEST DISTANCE IS $d\left(\frac{8}{3}, \frac{2}{3}\right) = \sqrt{\left(\frac{8}{3} - 2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(4 - \frac{10}{3}\right)^2}$

$$= \sqrt{3\left(\frac{2}{3}\right)^2} = \boxed{\sqrt{\frac{4}{3}}}$$

35. $x + y + z = 100 \Rightarrow z = 100 - x - y$

$P(x,y) = xy(100 - x - y) = 100xy - x^2y - xy^2$ ← MAXIMIZE P.

$P_x = 100y - 2xy - y^2 = 0$ $\quad$ $P_y = 100x - x^2 - 2xy = 0$

$y(100 - 2x - y) = 0$ $\quad$ IF $\underline{y = 0}$: $x(100 - x) = 0$

$\swarrow \quad \searrow$ $\qquad$ $\swarrow \quad \searrow$

$y = 0$ OR $y = 100 - 2x$ $\qquad$ $\underline{x = 0 \text{ OR } x = 100}$

IF $y = 100 - 2x$: $100x - x^2 - 2x(100 - 2x) = 0$

$100x - x^2 - 200x + 4x^2 = 0$

$3x^2 - 100x = x(3x - 100) = 0$

$\swarrow \quad \searrow$

$\frac{x = 0}{y = 100}$ OR $\frac{x = \frac{100}{3}}{y = \frac{100}{3}}$

CRITICAL PNTS: $(0,0), (100,0), (0,100), \left(\frac{100}{3}, \frac{100}{3}\right)$

$P_{xx} = -2y$ $\qquad$ $P_{yy} = -2x$ $\qquad$ $P_{xy} = 100 - 2x - 2y$

$D(0,0) = (0)(0) - (100)^2 < 0$ → SADDLE PNT AT $(0,0)$

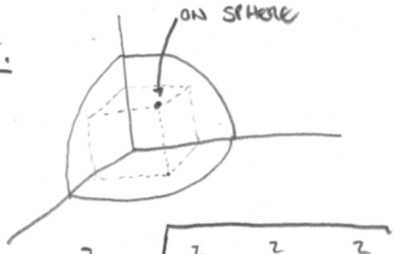
$D(100,0) = (-200)(0) - (-100)^2 < 0$ → SADDLE PNT AT $(100,0)$

$D(0,100) = (-200)(0) - (-100)^2 < 0$ → SADDLE PNT AT $(0,100)$

$D\left(\frac{100}{3}, \frac{100}{3}\right) = \left(-\frac{200}{3}\right)\left(-\frac{200}{3}\right) - \left(-\frac{100}{3}\right)^2 > 0$

$P_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) < 0$ → LOCAL MAX AT $\boxed{x = \frac{100}{3}, y = \frac{100}{3}, z = \frac{100}{3}}$

37.



BY SYMMETRY, WE WILL MAXIMIZE BOX IS FIRST OCTANT WITH OPPOSITE CORNERS AT ORIGIN & ON SPHERE, THEN MULTIPLY VOLUME BY 8.

$$z = \sqrt{r^2 - x^2 - y^2}$$

$$V(x,y) = xy\sqrt{r^2 - x^2 - y^2}$$ MAXIMIZE

INSTEAD WE WILL MAXIMIZE $V^2 = x^2y^2(r^2 - x^2 - y^2) = r^2x^2y^2 - x^4y^2 - x^2y^4$

$$V_x = 2r^2xy^2 - 4x^3y^2 - 2xy^4 = 0$$

$$2xy^2(r^2 - 2x^2 - y^2) = 0$$

$$\underbrace{\underline{x=0}, \ y=\underline{0}}, \text{ or } 2x^2 + y^2 = r^2$$

$$\underline{x^2 \overset{\sim}{=} \tfrac{1}{2}(r^2 - y^2)}$$

NOTE THAT IF x OR $y = 0$ THEN $V = 0$. ↳ NOT MAXIMUM.

$$V_y = 2r^2x^2y - 2x^4y - 4x^2y^3 = 0$$

→ SET $x^2 = \frac{1}{2}(r^2 - y^2)$

$$2x^2y(r^2 - x^2 - 2y^2) = 0$$

$$(r^2 - y^2)y\left[r^2 - \tfrac{1}{2}(r^2 - y^2) - 2y^2\right] = 0$$

$$(r^2 - y^2)y\left(\tfrac{1}{2}r^2 - \tfrac{3}{2}y^2\right) = 0$$

IF $r^2 = y^2$ THE $x = z = 0$ & $V = 0$

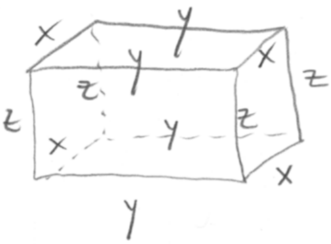
IF $y = 0$, $V = 0$

$r^2 = 3y^2$ $\Rightarrow \underline{\underline{y = \frac{r}{\sqrt{3}}}}$

THEN $x^2 = \frac{1}{2}\left(r^2 - \frac{r^2}{3}\right) = \frac{r}{\sqrt{3}}$ & $z = \sqrt{r^2 - \frac{r^2}{3} - \frac{r^2}{3}} = \frac{r}{\sqrt{3}}$

$$\& \quad \text{Vol} = \left(\frac{r}{\sqrt{3}}\right)^3 = \frac{r^3}{3\sqrt{3}} \xrightarrow{\times 8} \boxed{\frac{8r^3}{3\sqrt{3}}}$$

41.



$$4x + 4y + 4z = c$$

$$x + y + z = \frac{c}{4}$$

$$\underline{\underline{z = \frac{c}{4} - x - y}}$$

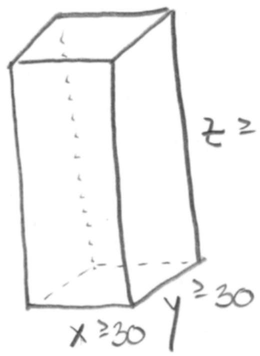
$$V(x,y) = xy\left(\frac{c}{4} - x - y\right) = \frac{c}{4}xy - x^2y - xy^2$$

↑

THIS IS THE SAME AS QUESTION #35 EXEPT 100 REPRACED BY $\frac{c}{4}$.

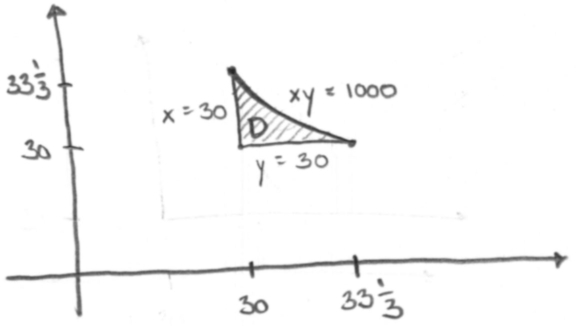
$$\therefore \boxed{(x, y, z) = \left(\frac{c}{12}, \frac{c}{12}, \frac{c}{12}\right)}$$

44.



$$z = \frac{4000}{xy} \geq 4 \rightarrow xy \leq 1000$$

(a) DOMAIN



$$\text{(b)} \quad H(x,y) = \overset{\text{NORTH/SOUTH}}{2x\left(\frac{4000}{xy}\right)(8)} + \overset{\text{EAST/WEST}}{2y\left(\frac{4000}{xy}\right)(10)} + \overset{\text{FLOOR}}{xy} + \overset{\text{ROOF}}{5xy}$$

$$= \frac{64000}{y} + \frac{80000}{x} + 6xy$$

MINIMIZE THIS OVER D

$H_x = -\frac{80\,000}{x^2} + 6y = 0$ $\qquad$ $H_y = -\frac{64\,000}{y^2} + 6x = 0$

$6x^2y = 80000$ $\qquad$ $6xy^2 = 64000$

$y = \frac{40000}{3x^2} \longrightarrow 6x\left(\frac{40000}{x^2}\right)^2 = 64000$

$$\frac{9600000000}{x^3} = 64000$$

$$150\,000 = x^3$$

$$x = (150000)^{1/3} = 10\sqrt[3]{150} \approx 53.13$$

NOT IN DOMAIN !

THEN $y = \frac{40000}{3(10\sqrt[3]{150})^2} \approx 4.72$, $z = \frac{4000}{xy} \approx 15.94$

NOT IN DOMAIN

BOUNDARY : $x = 30$: $H(30,y) = \frac{64000}{y} + \frac{8000}{3} + 180y$, $y \in [30, 33\frac{1}{3}]$

$H'(y) = -\frac{64000}{y^2} + 180 = 0$

$$180y^2 = 64000$$

$$y \approx 18.86$$

NOT IN DOMAIN

$y = 30$: $H(x,30) = \frac{6400}{3} + \frac{80000}{x} + 180x$, $x \in [30, 33\frac{1}{3}]$

$H'(x) = -\frac{80000}{x^2} + 180 = 0$

$$180x^2 = 80000$$

$$x \approx 21$$

NOT IN DOMAIN

11.7.10

$$xy = 1000 \longrightarrow y = \frac{1000}{x}$$

$$H\left(x, \frac{1000}{x}\right) = 64x + \frac{80000}{x} + 6000$$

$$H'(x) = 64 - \frac{80000}{x^2} = 0$$

$$64x^2 = 80000$$

$$x \approx 35.36$$

NOT IN DOMAIN.

∴ MINIMUM VALUE OF $H(x,y)$ MUST OCCUR AT ONE OF THE CORNERS

$$(30,30),\ (30, 33\tfrac{1}{3}),\ (33\tfrac{1}{3}, 30).$$

$$H(30,30) = \frac{64000}{30} + \frac{80000}{30} + 6(30)(30) = 10{,}200$$

$$H\left(30, \frac{100}{3}\right) = \frac{64000}{30} + \frac{3 \cdot 80000}{100} + 6(30)\left(\frac{100}{3}\right) = 10{,}533\tfrac{1}{3}$$

$$H\left(\frac{100}{3}, 30\right) = \frac{3 \cdot 64000}{100} + \frac{80000}{30} + 6\left(\frac{100}{3}\right)(30) = 10{,}506\tfrac{2}{3}$$

MIN AT $x = 30$, $y = 30$, $z = \dfrac{4000}{(30)(30)} = 4\tfrac{4}{9}$

MIN HEAT LOSS = 10,200 UNITS/DAY

(c) YES. ALL OF OUR CRITICAL PNTS IN PART (b) WERE OUTSIDE OF THE DOMAIN. LIKELY, AT LEAST ONE WOULD BE A MINIMUM (THOUGH WE WOULD HAVE TO CHECK THAT WITH 2nd DERIV'S TEST.)

11.7.11