§ 12.3 Double Integrals in Polar Coord.

John Adamski
11/18/2014

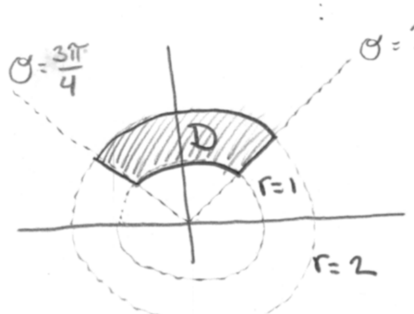
1. Polar : $$\int_0^{3\pi/2}\int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

2. Rectangular : $$\int_{-1}^{1}\int_0^{1-x^2} f(x,y)\, dy\, dx$$

3. Rectangular : $$\int_{-1}^{1}\int_0^{\frac{1}{2}(1+x)} f(x,y)\, dy\, dx$$

4. Polar : $$\int_{-\pi/2}^{\pi/2}\int_3^6 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

5.



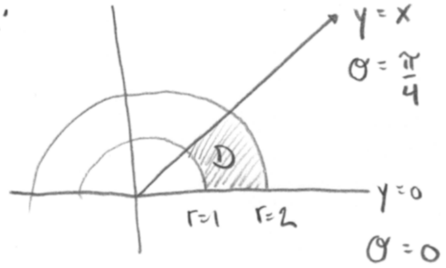
$$\int_{\pi/4}^{3\pi/4}\int_1^2 r\, dr\, d\theta = \int_{\pi/4}^{3\pi/4} \frac{1}{2} r^2 \Big|_1^2 \, d\theta$$

$$= \frac{3}{2}\int_{\pi/4}^{3\pi/4} d\theta = \boxed{\frac{3\pi}{4}}$$

10. $$\int_0^{2\pi}\int_a^b r\sin^2\theta\, dr\, d\theta = \frac{b^2-a^2}{2}\int_0^{2\pi}\sin^2\theta\, d\theta$$

$$= \frac{b^2-a^2}{4}\int_0^{2\pi} 1 - \underbrace{\cos 2\theta}_{0}\, d\theta = \boxed{\frac{\pi(b^2-a^2)}{2}}$$

11.



$$\int_0^{\pi/4}\int_1^2 \underbrace{\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right)} r\,dr\,d\theta$$

$$= \tan^{-1}(\tan(\theta))$$

$$= \theta$$

$$= \int_0^{\pi/4}\int_1^2 r\theta\,dr\,d\theta = \frac{1}{2}(2^2-1^2)\int_0^{\pi/4} d\theta = \boxed{\frac{3\pi}{8}}$$

12.

$$\int_0^{2\pi}\int_0^2 r\cos(r)\,dr\,d\theta$$

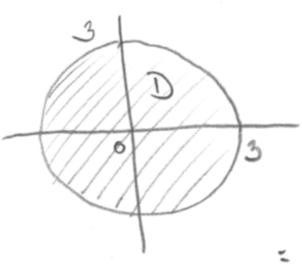
LET $u = r$ $\quad v = \sin(r)$

$du = dr$ $\quad dv = \cos(r)\,dr$

$$= \int_0^{2\pi}\left(r\sin(r)\Big|_0^2 - \int_0^2 \sin(r)\,dr\right) d\theta$$

$$= \int_0^{2\pi}\left(2\sin(2) + \cos(2) - 1\right) d\theta = \boxed{2\pi\left(2\sin(2)+\cos(2)-1\right)}$$

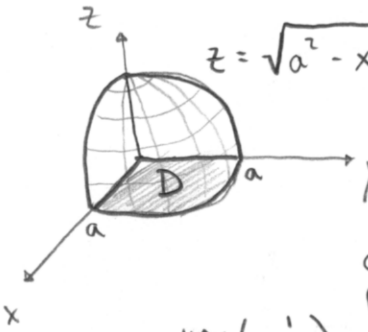
14. INTERSECTION OF PARABOLOID & XY-PLANE: $z = 18 - 2x^2 - 2y^2 = 0 \rightarrow x^2 + y^2 = 9$



$$\int_0^{2\pi}\int_0^3 (18 - 2r^2)\,r\,dr\,d\theta = 2\pi\int_0^3 18r - 2r^3\,dr$$

$$= 2\pi\left[9r^2 - \frac{1}{2}r^4\right]_0^3 = 2\pi\left(81 - \frac{81}{2}\right) = \boxed{81\pi}$$

15. LET US FIND VOL. OF PORTION IN 1^{st} OCTANT, THEN $\times 8$.



$z = \sqrt{a^2 - x^2 - y^2}$

$$V = 8 \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2} \cdot r \, dr \, d\theta$$

LET $u = a^2 - r^2$

$du = -2r \, dr$

$$= 4\pi \left(-\frac{1}{2}\right) \int_{a^2}^{0} \sqrt{u} \, du = 2\pi \cdot \frac{2}{3} u^{3/2} \Big|_0^{a^2} = \boxed{\frac{4\pi a^3}{3}}$$

17. INTERSECT: CONE: $z^2 = x^2 + y^2$
SPHERE: $z^2 = 1 - x^2 - y^2$

SET EQUAL: $x^2 + y^2 = 1 - x^2 - y^2$

$x^2 + y^2 = \frac{1}{2} \longrightarrow r = \frac{1}{\sqrt{2}}$

$$\int_0^{2\pi} \int_0^{1/\sqrt{2}} \Big(\underbrace{\sqrt{1 - r^2}}_{\substack{\text{TOP} \\ \text{(SPHERE)}}} - \underbrace{r}_{\substack{\text{BOT} \\ \text{(CONE)}}} \Big) r \, dr \, d\theta = 2\pi \left[-\frac{1}{3}(1 - r^2)^{3/2} - \frac{1}{3} r^3 \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= 2\pi \left[\left[-\frac{1}{3}\left(1 - \frac{1}{2}\right)^{3/2} - \frac{1}{3}\left(\frac{1}{2\sqrt{2}}\right) \right] - \left[-\frac{1}{3}(1)^{3/2} - 0 \right] \right]$$

$$= 2\pi \left[\left(\frac{-1}{6\sqrt{2}} - \frac{1}{6\sqrt{2}} \right) + \frac{1}{3} \right] = 2\pi \left(\frac{1}{3} - \frac{1}{3\sqrt{2}} \right) = 2\pi \left(\frac{\sqrt{2} - 1}{3\sqrt{2}} \right)$$

$$= \frac{\sqrt{2}\,\pi(\sqrt{2} - 1)}{3} = \boxed{\frac{\pi(2 - \sqrt{2})}{3}}$$

19. TAKE VOLUME IN 1^{st} OCTANT & $\times 8$

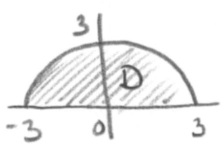
ELLIPSOID: $z^2 = 64 - 4(x^2+y^2)$

$$z = \sqrt{64 - 4(x^2+y^2)} \longrightarrow \sqrt{64-4r^2} = 2\sqrt{16-r^2}$$

$$V = 8\int_0^{\pi/2}\int_0^2 2\sqrt{16-r^2}\cdot r\,dr\,d\theta = 8\pi\int_0^2 r\sqrt{16-r^2}\,dr$$

$$= 4\pi\int_{12}^{16}\sqrt{u}\,du = 4\pi\cdot\frac{2}{3}u^{3/2}\Big|_{12}^{16} = \boxed{\frac{8\pi}{3}\left(64-24\sqrt{3}\right)}$$

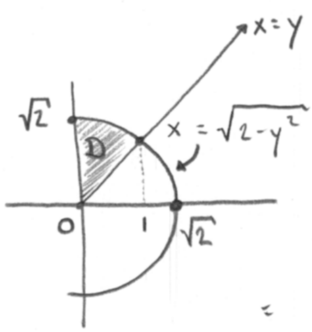
23.



$$\int_0^{\pi}\int_0^3 \sin(r^2)\,r\,dr\,d\theta = \frac{\pi}{2}\int_0^9 \sin(u)\,du$$

$$= \boxed{-\frac{\pi}{2}\sin(9)}$$

25.

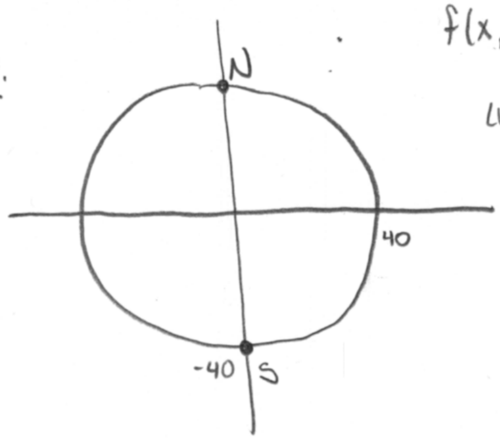


$$\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} r^2(\cos\theta + \sin\theta)\,dr\,d\theta$$

$$= \frac{1}{3}r^3\Big|_0^{\sqrt{2}}\int_{\pi/4}^{\pi/2}\cos\theta + \sin\theta\,d\theta = \frac{2\sqrt{2}}{3}\Big[\sin\theta - \cos\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{2\sqrt{2}}{3}\left[(1-0) - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] = \boxed{\frac{2\sqrt{2}}{3}}$$

27.



$f(x,y) = f(y)$ (DOES NOT DEPEND ON x)

LINEAR: $f(-40) = 2$, $f(40) = 7$

$f(x,y) = \frac{9}{2} + \frac{1}{16} y$ ← DEPTH AT (x,y)

$$\int_0^{2\pi}\int_0^{40} \left(\frac{9}{2} + \frac{1}{16} r\sin\theta\right) r \, dr \, d\theta = \int_0^{2\pi} \left[\frac{9}{4}r^2 + \frac{1}{48} r^3 \sin\theta\right]_{r=0}^{r=40} d\theta$$

$$= \int_0^{2\pi} \frac{9}{4}\cdot 40^2 + \underbrace{\frac{1}{48}\cdot 40^3 \sin\theta \, d\theta}_{\text{INTEGRAL} = 0} = \frac{9\pi}{2}\cdot 40^2 = \boxed{7200\pi}$$

28. (a) $\displaystyle\int_0^{2\pi}\int_0^{R} e^{-r}\cdot r \, dr \, d\theta$

LET $u = r$ $\quad v = -e^{-r}$

$du = dr$ $\quad dv = e^{-r} dr$

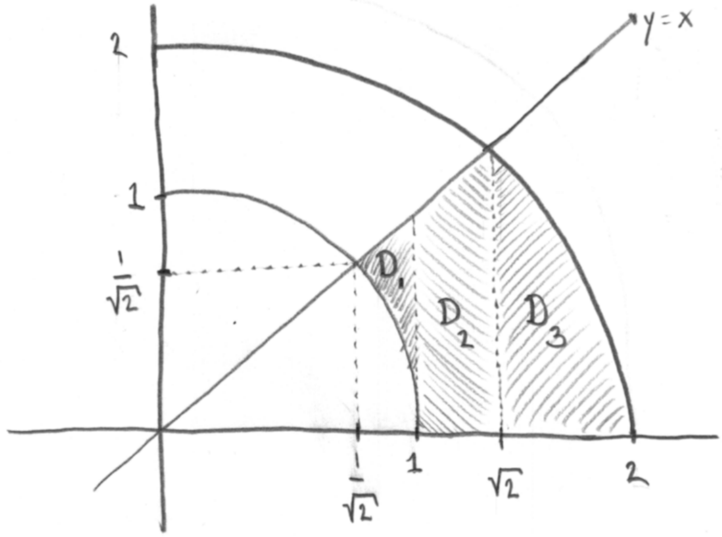
$$= 2\pi\left(-re^{-r}\Big|_0^R + \int_0^R e^{-r} dr\right) = 2\pi\left(-Re^{-R} - e^{-r}\Big|_0^R\right)$$

$$= 2\pi\left(-Re^{-R} - e^{-R} + 1\right) = 2\pi\left(\frac{-R-1}{e^R} + 1\right)$$

(b) TOTAL AREA $= \pi R^2$

$$\therefore \text{AVE} = \frac{2\cancel{\pi}\left(1 - \frac{R+1}{e^R}\right)}{\cancel{\pi} R^2} = \boxed{\frac{2}{R^2}\left(1 - \frac{R+1}{e^R}\right)}$$

29. $\underbrace{\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx}_{D_1} + \underbrace{\int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx}_{D_2} + \underbrace{\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx}_{D_3}$



$$\int_0^{\pi/4}\int_1^2 r^3 \underbrace{\cos\theta \sin\theta}_{=\frac{1}{2}\sin 2\theta}\,dr\,d\theta = \frac{1}{4}r^4\Big|_1^2 \cdot \frac{1}{2}\int_0^{\pi/4} \sin 2\theta\,d\theta$$

$$= \frac{15}{8} \cdot -\frac{1}{2}\cos 2\theta\Big|_0^{\pi/4} = \boxed{\frac{15}{16}}$$