§12.5 TRIPLE INTEGRALS

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7. $$\int_0^3\int_0^x\int_{x-y}^{x+y} y\,dz\,dy\,dx = \int_0^3\int_0^x 2y^2\,dy\,dx = \int_0^3 \frac{2}{3}x^3\,dx = \frac{1}{6}x^4\Big|_0^3 = \frac{81}{6} = \boxed{\frac{27}{2}}$$

$$z\Big|_{x-y}^{x+y} = x+y-x+y = 2y$$

9. $$\int_1^4\int_y^4\int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy = \int_1^4\int_y^4 \frac{1}{z}\tan^{-1}\left(\frac{x}{z}\right)\Big|_0^z\,dz\,dy$$

$$= \int_1^4\int_y^4 \frac{\pi}{4}z\,dz\,dy = \frac{\pi}{8}\int_1^4 16-y^2\,dy = \frac{\pi}{8}\left(16y-\frac{1}{3}y^3\right)\Big|_1^4$$

$$= \frac{\pi}{8}\left(48-21\right) = \boxed{\frac{27\pi}{8}}$$

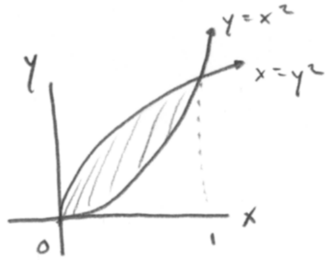
11. $$\int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xy\,dz\,dy\,dx = 6\int_0^1\int_0^{\sqrt{x}} xy+x^2y+xy^2\,dy\,dx$$

$$6xy(1+x+y) = 6(xy+x^2y+xy^2)$$

$$= 6\int_0^1 (x+x^2)\frac{1}{2}y^2\Big|_0^{\sqrt{x}} + \frac{1}{3}xy^3\Big|_0^{\sqrt{x}}\,dx = 6\int_0^1 \frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{3}x^{5/2}\,dx$$

$$= 6\left(\frac{1}{6}+\frac{1}{8}+\frac{2}{21}\right) = 6\cdot\frac{65}{168} = \boxed{\frac{65}{28}}$$

12.



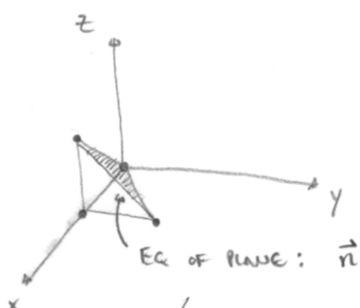
$$\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{x+y} xy \, dz\, dy\, dx = \int_0^1\int_{x^2}^{\sqrt{x}} x^2y + xy^2 \, dy\, dx$$

$$= \int_0^1 \frac{1}{2}x^2y^2 + \frac{1}{3}xy^3 \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 \frac{1}{2}x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^{5/2} - \frac{1}{3}x^7 \, dx$$

$$\frac{1}{2}x^2(x - x^4) + \frac{1}{3}x(x^{3/2} - x^6)$$

$$= \frac{1}{8} - \frac{1}{14} + \frac{2}{21} - \frac{1}{24} = \frac{21 - 12 + 16 - 7}{168} = \frac{18}{168} = \frac{9}{84} = \boxed{\frac{3}{28}}$$

14.



$$\int_0^1\int_0^x\int_0^{x-y} xyz \, dz\, dy\, dx = \frac{1}{2}\int_0^1\int_0^x xy(x-y)^2 \, dy\, dx$$

$$xy(x^2 - 2xy + y^2)$$

EQ OF PLANE: $\vec{n} = \langle 1,0,1\rangle \times \langle 1,1,0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, 1, 1\rangle$

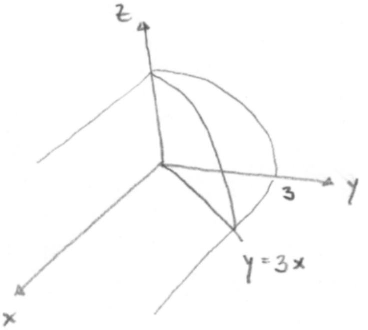
$\hookrightarrow -x + y + z = 0$

$z = x - y$

$$= \frac{1}{2}\int_0^1\int_0^x x^3y - 2x^2y^2 + xy^3 \, dy\, dx = \frac{1}{2}\int_0^1 \frac{1}{2}x^5 - \frac{2}{3}x^5 + \frac{1}{4}x^5 \, dx = \frac{1}{2}\int_0^1 \frac{1}{8}x^5 dx$$

$$= \frac{1}{2}\cdot\frac{1}{48}x^6\Big|_0^1 = \boxed{\frac{1}{96}}$$

16.



$$\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = \frac{1}{2}\int_0^1\int_{3x}^3 9-y^2\,dy\,dx$$

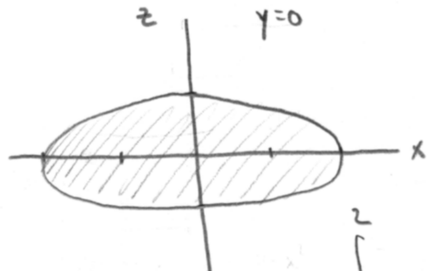
$$9(3-3x)-\frac{1}{3}\left(3^3-(3x)^3\right)$$

$$27-27x-9+9x^3$$

$$=\frac{1}{2}\int_0^1 18-27x+9x^3\,dx = \frac{1}{2}\left(18-\frac{27}{2}+\frac{9}{4}\right)$$

$$=\frac{1}{2}\left(18-\frac{45}{4}\right)=\frac{1}{2}\left(\frac{72-45}{4}\right)=\boxed{\frac{27}{8}}$$

27.

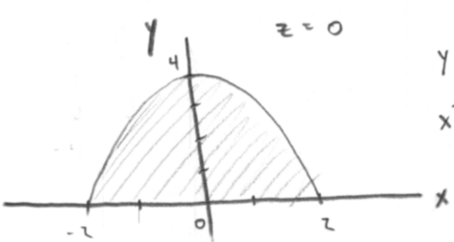


$$x^2+4z^2=4 \longrightarrow x=\pm\sqrt{4-4z^2}$$

$$\frac{x^2}{2^2}+z^2=1 \longrightarrow z=\pm\sqrt{1-\frac{1}{4}z^2}$$

$$\int_{-2}^{2}\int_{-\sqrt{1-\frac{1}{4}z^2}}^{\sqrt{1-\frac{1}{4}z^2}}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dz\,dx$$

$$\int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dx\,dz$$



$$y=4-x^2$$

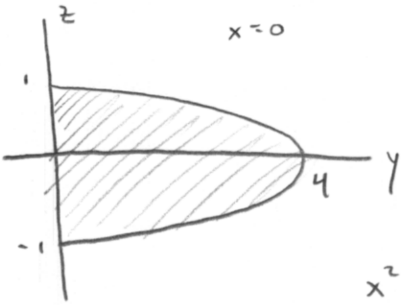
$$x^2=4-y \rightarrow x=\pm\sqrt{4-y}$$

$$4z^2=4-x^2-y$$

$$z^2=1-\frac{1}{4}x^2-\frac{1}{4}y \longrightarrow z=\pm\sqrt{1-\frac{1}{4}x^2-\frac{1}{4}y}$$

$$\int_{-2}^{2}\int_0^{4-x^2}\int_{-\sqrt{1-\frac{1}{4}x^2-\frac{1}{4}y}}^{\sqrt{1-\frac{1}{4}x^2-\frac{1}{4}y}} f(x,y,z)\,dz\,dy\,dx$$

$$\int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{1-\frac{1}{4}x^2-\frac{1}{4}y}}^{\sqrt{1-\frac{1}{4}x^2-\frac{1}{4}y}} f(x,y,z)\,dz\,dx\,dy$$



$x=0$

$y = 4-4z^2$

$4z^2 = 4-y \rightarrow z^2 = 1-\frac{1}{4}y$

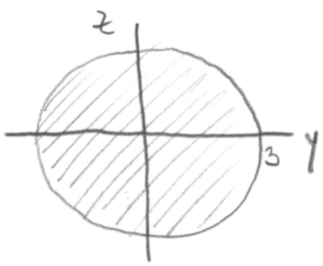
$z = \pm\sqrt{1-\frac{1}{4}y}$

$x^2 = 4-4z^2-y \longrightarrow x = \pm\sqrt{4-4z^2-y}$

$$\int_0^4 \int_{-\sqrt{1-\frac{1}{4}y}}^{\sqrt{1-\frac{1}{4}y}} \int_{-\sqrt{4-4z^2-y}}^{\sqrt{4-4z^2-y}} f(x,y,z)\,dx\,dz\,dy$$

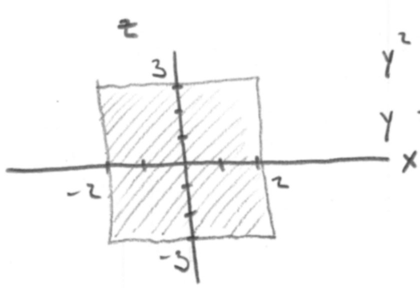
$$\int_{-1}^{1} \int_0^{4-4z^2} \int_{-\sqrt{4-4z^2-y}}^{\sqrt{4-4z^2-y}} f(x,y,z)\,dx\,dy\,dz$$

<u>28.</u> $y^2+z^2=9$, $x=-2$, $x=2$



$$\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-2}^{2} f(x,y,z)\,dx\,dz\,dy$$

$$\int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{-2}^{2} f(x,y,z)\,dx\,dy\,dz$$

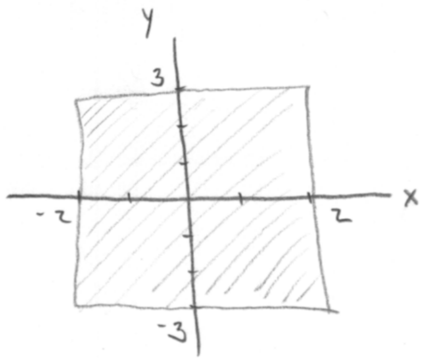


$y^2 = 9-z^2$

$y = \pm\sqrt{9-z^2}$

$$\int_{-2}^{2} \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dz\,dx$$

$$\int_{-3}^{3} \int_{-2}^{2} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dx\,dz$$



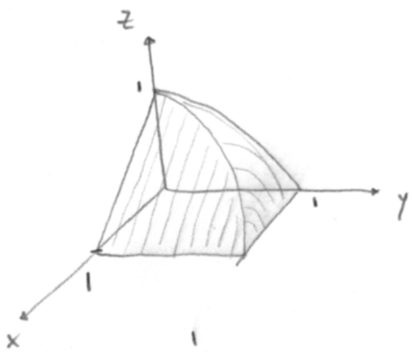
$$z^2 = 9 - y^2$$

$$z = \pm\sqrt{9 - y^2}$$

$$\int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dy\,dx$$

$$\int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dx\,dy$$

<u>38.</u>



$x = 1 - z$ OVER $\{(y,z) : 0 \le z \le 1 - y^2,\ 0 \le y \le 1\}$

$$m = \int_0^1\int_0^{1-y^2}\int_0^{1-z} 4\,dx\,dz\,dy = \int_0^1\int_0^{1-y^2} 4 - 4z\,dz\,dy$$

$$= \int_0^1 4 - 4y^2 - 2(1-y^2)^2\,dy = 2y - 2y^2 + \frac{4}{3}y^3 - \frac{2}{5}y^5 \Big|_0^1$$

$$4 - 4y - 2 + 4y^2 - 2y^4$$

$$= \frac{4}{3} - \frac{2}{5} = \frac{20-6}{15} = \boxed{\frac{14}{15}}$$

$$\bar{x} = \frac{15}{14}\int_0^1\int_0^{1-y^2}\int_0^{1-z} 4x\,dx\,dz\,dy = \frac{15}{7}\int_0^1\int_0^{1-y^2} 1 - 2z + z^2\,dz\,dy$$

$$= \frac{15}{7}\int_0^1 z - z^2 + \frac{1}{3}z^3 \Big|_0^{1-y^2} = \frac{5}{7}\int_0^1 1 - y^6\,dy = \frac{5}{7}\left(1 - \frac{1}{7}\right) = \boxed{\frac{30}{49}}$$

$$1 - y^2 - (1-y^2)^2 + \frac{1}{3}(1-y^2)^3$$

$$1 - y^2 - (1 - 2y^2 + y^4) + \frac{1}{3}(1 - 3y^2 + 3y^4 - y^6)$$

$$\cancel{1} - \cancel{y^2} - \cancel{1} + \cancel{2y^2} - \cancel{y^4} + \frac{1}{3} - \cancel{y^2} + \cancel{y^4} - \frac{1}{3}y^6$$

$$\bar{y} = \frac{15}{7}\int_0^1\int_0^{1-y^2}\int_0^{1-z} 2y\,dx\,dz\,dy = \frac{15}{7}\int_0^1\int_0^{1-y^2} 2y - 2yz\,dz\,dy$$

$$2y(1-z) = 2y - 2yz$$

$$2y(1-y^2) - y(1-y^2)^2 = 2y - \cancel{2y^3} - y + \cancel{2y^3} - y^5$$

$$= \frac{15}{7}\int_0^1 y - y^5\,dy = \frac{15}{7}\left(\frac{1}{2} - \frac{1}{6}\right) = \boxed{\frac{5}{7}}$$

$$\bar{z} = \frac{15}{7}\int_0^1\int_0^{1-y^2}\int_0^{1-z} 2z\,dx\,dz\,dy = \frac{15}{7}\int_0^1\int_0^{1-y^2} 2z - 2z^2\,dz\,dy$$

$$2z(1-z) = 2z - 2z^2$$

$$(1-y^2)^2 - \frac{2}{3}(1-y^2)^3 = 1 - 2y^2 + y^4 - \frac{2}{3}(1 - 3y^2 + 3y^4 - y^6)$$

$$= 1 - \cancel{2y^2} + y^4 - \frac{2}{3} + \cancel{2y^2} - 2y^4 + \frac{2}{3}y^6$$

$$= \frac{15}{7}\int_0^1 \frac{1}{3} - y^4 + \frac{2}{3}y^6\,dy = \frac{15}{7}\left(\frac{1}{3} - \frac{1}{5} + \frac{2}{21}\right) = \frac{15}{7}\left(\frac{35-21+10}{105}\right)$$

$$= \frac{\cancel{15}}{7}\cdot\frac{24}{\cancel{105}_7} = \boxed{\frac{24}{49}}$$

39. $$m = \int_0^a\int_0^a\int_0^a x^2+y^2+z^2\,dz\,dy\,dx = \int_0^a\int_0^a ax^2+ay^2+\frac{1}{3}a^3\,dy\,dx$$

$$= \int_0^a a^2x^2 + \underbrace{\frac{1}{3}a^4 + \frac{1}{3}a^4}_{\frac{2}{3}a^4}\,dx = \frac{1}{3}a^5 + \frac{2}{3}a^5 = a^5$$

BY SYMMETRY, $\bar{x} = \bar{y} = \bar{z} = \frac{1}{a^5} \int_0^a \int_0^a \int_0^a x^2 z + y^2 z + z^3 \, dz \, dy \, dx$

$\frac{1}{2}x^2 z^2 + \frac{1}{2} y^2 z^2 + \frac{1}{4} z^4 \Big|_0^a$

$= \frac{1}{4a^3} \int_0^a \int_0^a 2x^2 + 2y^2 + a^2 \, dy \, dx$

$2ax^2 + \frac{2}{3}a^3 + a^3$

$= \frac{1}{4a^2} \int_0^a 2x^2 + \frac{5}{3}a^2 \, dx = \frac{1}{4a^2}\left(\frac{2}{3}a^3 + \frac{5}{3}a^3\right) = \boxed{\frac{7a}{12}}$

47. $I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\, dV$

$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h x^2 + y^2 \, dz \, dy \, dx = h \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2 + y^2 \, dy \, dx$

$\rightsquigarrow h \int_0^{2\pi} \int_0^a r^3 \, dr \, d\theta = 2\pi h \cdot \frac{1}{4} r^4 \Big|_{r=0}^{r=a} = \boxed{\frac{\pi h}{2} a^4}$ ASSUMING DENSITY $\equiv 1$.

49. $V = L^3$ (OBVIOUSLY)

$f_{AVE} = \frac{1}{L^3} \int_0^L \int_0^L \int_0^L xyz \, dz \, dy \, dx = \frac{1}{2L} \int_0^L \int_0^L xy \, dy \, dx$

$= \frac{L}{4} \int_0^L x \, dx = \boxed{\frac{L^3}{8}}$