Name: * ANSWER KEY * 10/12/2017

Math 203 Calculus III Exam 1

1. (8 points) Find equations of the spheres with center $(1, -2, 3)$ that touch
   (a) the plane $x = -1$;

$$\boxed{(x-1)^2 + (y+2)^2 + (z-3)^2 = 4}$$

   (b) the origin.

$$r = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\boxed{(x-1)^2 + (y+2)^2 + (z-3)^2 = 14}$$

2. (6 points) Let **a**, **b**, **c**, and **d** be vectors. State whether each of the following expressions is meaningful or not. If yes, state whether the result is a scalar or a vector.

   (a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ SCALAR

   (b) $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ No

   (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ VECTOR

   (d) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ No

   (e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ No

   (f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ SCALAR

3. (8 points) Consider the two planes P_1 and P_2.

$$\begin{aligned} P_1: &\quad 2x - 4y + 5z = 2 \\ P_2: &\quad 5x + 2y - 4z = 5 \end{aligned}$$

(a) Find the angle at which the two planes intersect.

ANGLE BETWEEN NORMAL VECTORS :

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$$

$$\langle 2,-4,5\rangle \cdot \langle 5,2,-4\rangle = \sqrt{2^2+4^2+5^2}\,\sqrt{5^2+2^2+4^2}\,\cos\theta$$

$$10 - 8 - 20 = 45\cos\theta$$

$$\cos\theta = -\frac{18}{45} \longrightarrow \boxed{\theta = \cos^{-1}\left(-\frac{2}{5}\right)}$$

(b) Give either a vector equation or parametric equations for the line of intersection of P_1 and P_2.
Hint: It is easy to see that one point on the line of intersection is $(1,0,0)$.

THE LINE OF INTERSECTION LIES IN BOTH PLANES

$\Rightarrow$ IS $\perp$ TO BOTH NORMAL VECTORS.

$$\Rightarrow \text{ IS } \parallel \text{ TO } \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 5 & 2 & -4 \end{vmatrix} = \langle 6, 33, 24\rangle$$

NOTE: ANY SCALAR MULTIPLE OF THIS VECTOR WILL DO.

$$\boxed{\begin{aligned} &\vec{\ell}(t) = \langle 1,0,0\rangle + t\langle 6,33,24\rangle \\ &\text{OR} \quad x = 1 + 6t \\ &\qquad\ \ y = 33t \\ &\qquad\ \ z = 24t \end{aligned}}$$

4. (4 points) Give a vector equation and parametric equations for the line *segment* from $(9, -2, 5)$ to $(7, 3, -6)$.

VECTOR: $\vec{r}(t) = (1-t)\langle 9, -2, 5\rangle + t\langle 7, 3, -6\rangle$, $0 \le t \le 1$

OR $\vec{r}(t) = \langle 9, -2, 5\rangle + t\langle -2, 5, -11\rangle$, $0 \le t \le 1$

PARAMETRIC: $x = 9 - 2t$

$y = -2 + 5t$, $0 \le t \le 1$

$z = 5 - 11t$

OTHERWISE, IT IS NOT A LINE SEGMENT

5. (8 points) Suppose a line l is parametrized by the equations

$$x = 4 + t, \qquad y = 3 - 2t, \qquad z = t - 7,$$

and a plane P is described by the equation

$$2x - 6y + z = 10.$$

Give parametric equations for the line through the point $(5, 8, 5)$ that is parallel to the plane P and perpendicular to the line l.

LINE IS $\parallel$ TO $\langle 1, -2, 1\rangle \times \langle 2, -6, 1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 2 & -6 & 1 \end{vmatrix}$

$= \langle 4, 1, -2\rangle$

$\vec{l}(t) = \langle 5, 8, 5\rangle + t\langle 4, 1, -2\rangle$

OR

$$\boxed{\begin{aligned} x &= 5 + 4t \\ y &= 8 + t \\ z &= 5 - 2t \end{aligned}}$$

A B C

6. (8 points) Give an equation for the plane that contains the three points $(2,1,2)$, $(9,1,7)$ and $(6,4,6)$.

PLANE IS PARALLEL TO VECTORS $\vec{AB}$ & $\vec{AC}$

$\Rightarrow$ PLANE IS $\perp$ TO $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 0 & 5 \\ 4 & 3 & 4 \end{vmatrix} = \langle -15, -8, 21 \rangle$

$\therefore$ PLANE EQ: $-15x - 8y + 21z = d$ FIND d.

SINCE $(2,1,2)$ IS ON THE PLANE,

$-15(2) - 8(1) + 21(2) = d$

$-30 - 8 + 42 = d \Rightarrow d = 4$

$\boxed{-15x - 8y + 21z = 4}$

7. (4 points) Sketch the domain of the function $f(x,y) = \arcsin(x^2 + y^2 - 3)$.

DOMAIN OF ARCSIN = RANGE OF SIN = $[-1, 1]$

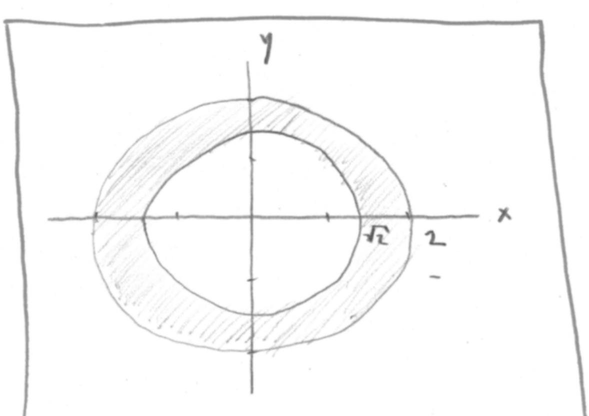
$\therefore -1 \le x^2 + y^2 - 3 \le 1$

$2 \le x^2 + y^2 \le 4$

OUTSIDE CIRCLE OF RADIUS $\sqrt{2}$

INSIDE CIRCLE OF RADIUS 2

BOTH CENTERED AT THE ORIGIN.



8. (10 points) Let C be the space curve described by the vector-valued function

$$\mathbf{r}(t) = \langle \cos t, 3\sin t, t \rangle, \qquad 0 \le t \le 6\pi$$

(a) Give parametric equations for the tangent line to C at the point $(0, 3, 5\pi/2)$.

NOTE THAT $\vec{r}\left(\frac{5\pi}{2}\right) = \langle 0, 3, \frac{5\pi}{2} \rangle$

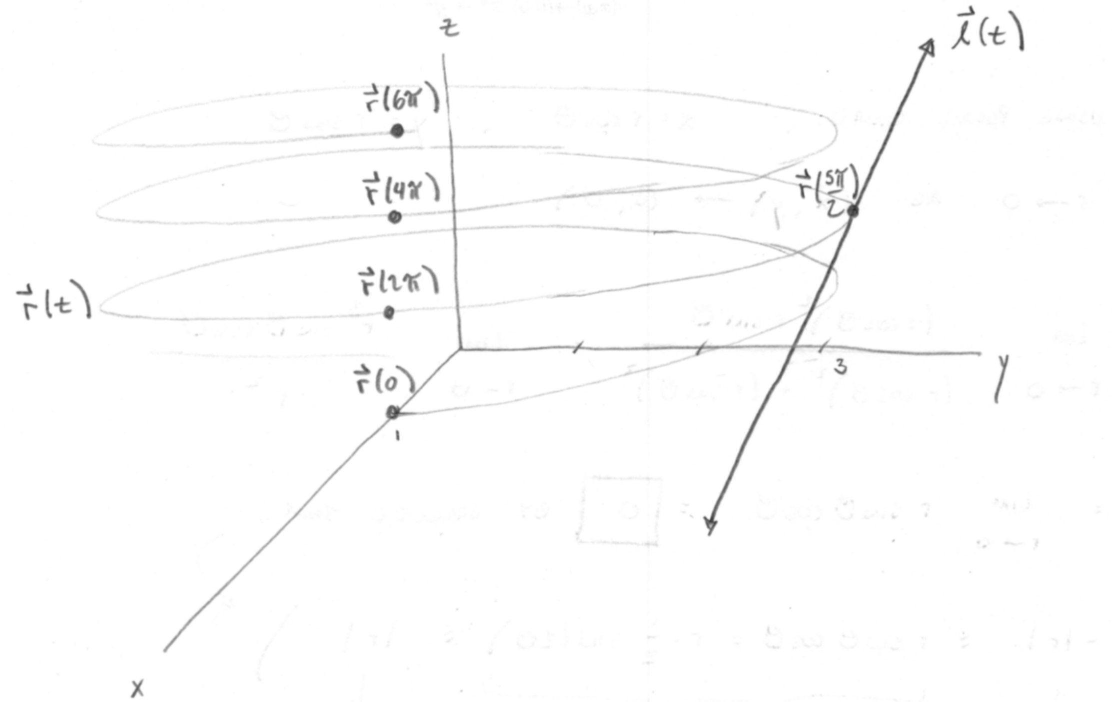
$\therefore \; \vec{\ell}(t) = \langle 0, 3, \frac{5\pi}{2} \rangle + t\,\vec{r}\,'\left(\frac{5\pi}{2}\right)$

$\vec{r}\,'(t) = \langle -\sin t, \; 3\cos t, \; 1 \rangle$

$\vec{r}\,'\left(\frac{5\pi}{2}\right) = \langle -1, 0, 1 \rangle$

$\therefore \; \vec{\ell}(t) = \langle 0, 3, \frac{5\pi}{2} \rangle + t\langle -1, 0, 1 \rangle \Rightarrow$
$\boxed{\begin{aligned} x &= -t \\ y &= 3 \\ z &= \frac{5\pi}{2} + t \end{aligned}}$

(b) Sketch the curve C and the tangent line from part (a) below.



9. (8 points) (a) Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$$

* ALONG X-AXIS $(y=0)$: $\lim_{x\to 0} \frac{0}{x^2} = 0$ *

* ALONG LINE $y = x$, $\lim_{x\to 0} \frac{x^2}{x^2+x^2+x^2} = \frac{1}{3}$ *

SINCE THE LIMIT DEPENDS ON THE PATH THAT $(x,y)\to(0,0)$ TAKES, THE LIMIT DOES NOT EXIST.

(b) Show that the following limit does exist, and find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

USING POLAR COORD., $x = r\cos\theta$, $y = r\sin\theta$

$r \to 0$ AS $(x,y) \to (0,0)$

$$\lim_{r\to 0} \frac{(r\cos\theta)^2 r\sin\theta}{(r\cos\theta)^2+(r\sin\theta)^2} = \lim_{r\to 0} \frac{r^3 \sin\theta\cos\theta}{r^2}$$

$= \lim_{r\to 0} r\sin\theta\cos\theta = \boxed{0}$ BY SQUEEZE THM.

$$\left(\underset{\downarrow \atop 0}{-|r|} \le \underbrace{r\sin\theta\cos\theta = r\cdot\frac{1}{2}\sin(2\theta)}_{\downarrow \atop 0} \le \underset{\downarrow \atop 0}{|r|} \right)$$

10. (6 points) Match each of the following equations with its graph (labeled A-F) and its contour map (labeled I-VI).

(a) $z = \sin(xy)$ C II

(b) $z = e^x \cos y$ A IV

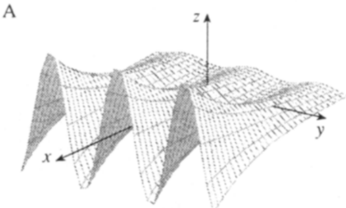
(c) $z = \sin(x - y)$ F I

(d) $z = \sin(x) - \sin(y)$ E III

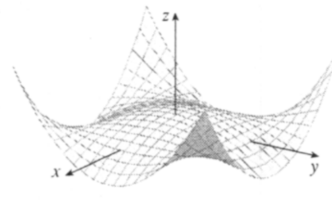
(e) $z = (1 - x^2)(1 - y^2)$ B VI

(f) $z = \dfrac{x - y}{1 + x^2 + y^2}$ D V

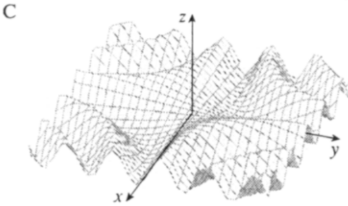
A



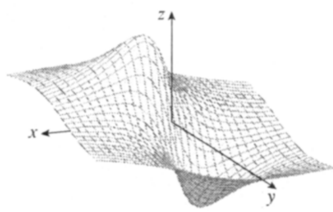
B



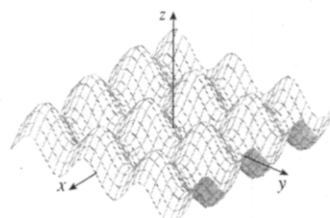
C



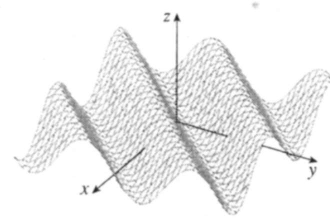
D



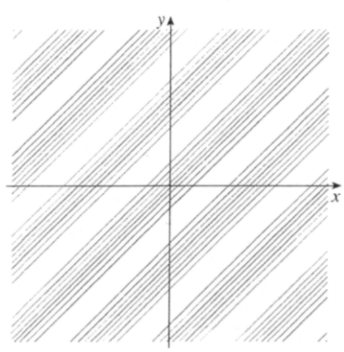
E



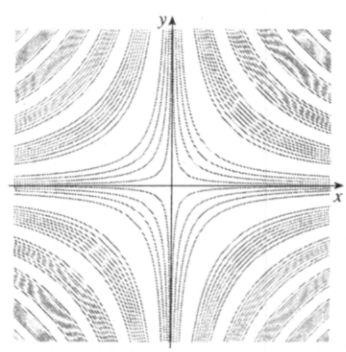
F



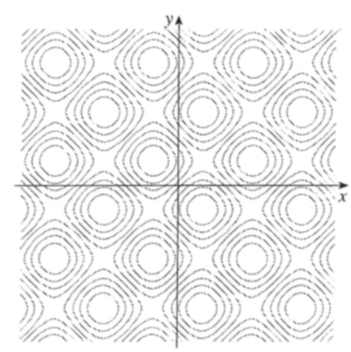
I



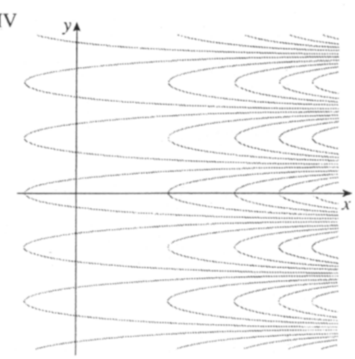
II



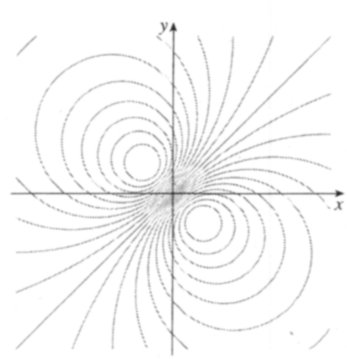
III



IV



V



VI