Name: *ANSWER KEY* 11/16/2017

Math 203 Calculus III **Exam 2**

1. (10 points) Let $f(x, y) = x\sin(x + y)$. Explain why f is differentiable at the point $(-1, 1)$. Then find an equation for the tangent plane to the graph $z = f(x, y)$ at the point $(-1, 1, 0)$.

$f_x = \sin(x+y) + x\cos(x+y)$

$f_y = x\cos(x+y)$

f IS DIFFERENTIABLE AT $(-1,1)$ BECAUSE BOTH f_x & f_y ARE CONTINUOUS ON AN OPEN SET CONTAINING $(-1,1)$, SPECIFICALLY $\mathbb{R}^2$ (ENTIRE PLANE).

TANGENT PLANE AT $(-1,1,0)$

$z = f(-1,1) + f_x(-1,1)(x+1) + f_y(-1,1,0)(y-1)$

$z = -1(x+1) - 1(y-1)$

$\boxed{z = -x - y}$

2. (8 points) Find all local maxima, local minima, and saddle points for

$$f(x, y) = 3x^2 - 12xy + 8y^3.$$

$f_x = 6x - 12y = 0$ $\qquad$ $f_y = -12x + 24y^2 = 0$

$x = 2y \xrightarrow{\text{SUBST.}}$ $\qquad$ $-12(2y) + 24y^2 = 0$

$-24y(1 - y) = 0$

$y = 0 \qquad y = 1$

$\downarrow \qquad \downarrow$

$x = 0 \qquad x = 2$

CRITICAL POINTS: $(0,0)$, $(1,2)$

$f_{xx} = 6 \qquad f_{yy} = 48y \qquad f_{xy} = f_{yx} = -12$

$D = f_{xx} f_{yy} - f_{xy}^2 = 288y - 144$

$(0,0)$: $D < 0 \Rightarrow$ $\boxed{(0,0) \text{ IS A SADDLE POINT}}$

$(1,2)$: $D > 0$ & $f_{xx} > 0 \Rightarrow$ $\boxed{(2,1) \text{ IS LOCAL MIN}}$

3. (12 points) At points (x, y, z) in a region of space for which $x^2 + y^2 \geq 1$ and $z \geq 0$, there is an electric charge

$$E(x, y, z) = z + z\ln(x^2 + y^2).$$

(a) Find the direction of greatest increase in E at $(1, 0, 2)$.

$$\nabla E(x,y,z) = \left\langle \frac{2xz}{x^2+y^2}, \frac{2yz}{x^2+y^2}, 1 + \ln(x^2+y^2) \right\rangle$$

$$\nabla E(1,0,2) = \boxed{\langle 4, 0, 1 \rangle}$$ OR ANY POSITIVE SCALAR MULTIPLE

(b) Find the rate at which the electric charge is changing at $(1, 0, 2)$ in the direction towards the point $(4, 4, 7)$.

$$\vec{v} = \text{FROM } (1,0,2) \text{ TO } (4,4,7) = \langle 3, 4, 5 \rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 3,4,5 \rangle}{\sqrt{3^2+4^2+5^2}} = \frac{1}{5\sqrt{2}} \langle 3, 4, 5 \rangle$$

$$D_{\vec{u}} f(1,0,2) = \nabla f(1,0,2) \cdot \vec{u} = \langle 4, 0, 1 \rangle \cdot \frac{1}{5\sqrt{2}} \langle 3, 4, 5 \rangle$$

$$= \boxed{\frac{1}{5\sqrt{2}} (12 + 5) = \frac{17}{5\sqrt{2}}}$$

(c) At each point (s, t) on the ground in a physics lab, the electric charge is measured at the corresponding point with the following coordinates.

$$x = s + t \qquad y = s - t \qquad z = 2st$$

Find the rate $\dfrac{\partial E}{\partial s}$ at which the electric charge is changing with respect to s at the point $(s, t) = (1, 1)$.

$$\frac{\partial E}{\partial s} = \frac{\partial E}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial E}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial E}{\partial z} \cdot \frac{\partial z}{\partial s}$$

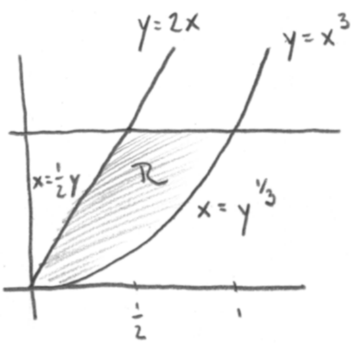
$$= \frac{2xz}{x^2+y^2} \cdot 1 + \frac{2yz}{x^2+y^2} \cdot 1 + \left(1 + \ln(x^2+y^2)\right) \cdot 2t$$

$$\therefore \left.\frac{\partial E}{\partial s}\right|_{\substack{s=1 \\ t=1}} \Rightarrow \begin{matrix} x = 2 \\ y = 0 \\ z = 2 \end{matrix} \qquad = 2 + 0 + 2(1 + \ln 4) = \boxed{4 + 2\ln 4}$$

4. (10 points) A laminar region $\mathcal{R}$ lies in the first quadrant, includes the origin, and is bounded by the following curves.

$$y = 2x \qquad y = 1 \qquad y = x^3$$

The density δ of the laminar at the point (x, y) is given by $\delta(x, y) = 24x^2$. Sketch the region $\mathcal{R}$ and compute its mass.



$$\text{MASS} \quad M = \iint_{\mathcal{R}} 24x^2 \, dA = \int_0^1 \int_{\frac{1}{2}y}^{y^{1/3}} 24x^2 \, dx \, dy$$

$$= \int_0^1 8x^3 \Big|_{x=\frac{1}{2}y}^{x=y^{1/3}} dy = \int_0^1 (8y - y^3) \, dy$$

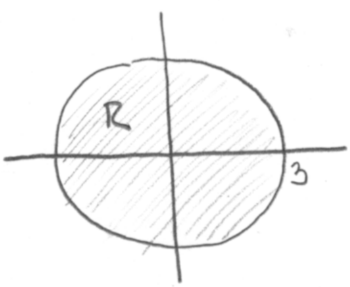
$$= 4y^2 - \frac{1}{4}y^4 \Big|_0^1 = 4 - \frac{1}{4} = \boxed{\frac{15}{4}}$$

5. (10 points) Find the volume of the solid that lies below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane.

$$\text{VOLUME} \quad V = \iint_R (18 - 2x^2 - 2y^2)\, dA$$

WHERE R IS THE INTERSECTION OF THE PARABOLOID $z = 18 - 2x^2 - 2y^2$ WITH THE XY-PLANE $z = 0$:

$$18 - 2x^2 - 2y^2 = 0 \longrightarrow x^2 + y^2 = 3^2$$



IN POLAR COORD :

$$R = \{(r, \theta) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$$

$$z = 18 - 2r^2$$

$$dA = r\, dr\, d\theta$$

$$\therefore \text{VOLUME} = \int_0^{2\pi}\int_0^3 (18 - 2r^2) r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^3 18r - 2r^3\, dr = 2\pi \left[9r^2 - \frac{1}{2} r^4 \right]_0^3$$

$$= 2\pi \left(81 - \frac{81}{2} \right) = \boxed{81\pi}$$