Extra Credit 1

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Problem

Find a function $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable everywhere, but f' is not continuous at 0. Show that this function satisfies these conditions.

Solution

Surely there are many such functions, but here is one.

$$f(x) = \begin{cases} x^2 \sin(1/x) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

It is clear that f is differentiable away from zero. To show that f is differentiable at 0, we need to go back to the definition of the derivative.

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} \\ &= \lim_{x \to 0} x \sin(1/x) \end{aligned}$$

Since

$$-1 \leqslant \sin(1/x) \leqslant 1,$$

it follows that

$$-|x| \leqslant x \sin(1/x) \leqslant |x|.$$

Thus,

$$0 = \lim_{x \to 0} -|x| \leqslant \lim_{x \to 0} x \sin(1/x) \leqslant \lim_{x \to 0} |x| = 0,$$

and so

$$f'(0) = 0.$$

Now we have

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

All the remains is to show that f' is not continuous at 0. But this is true since

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(2x\sin(1/x) - \cos(1/x)\right)$$

does not exist. To see this, just notice that the last term does not converge as $x \to 0$. $\square$

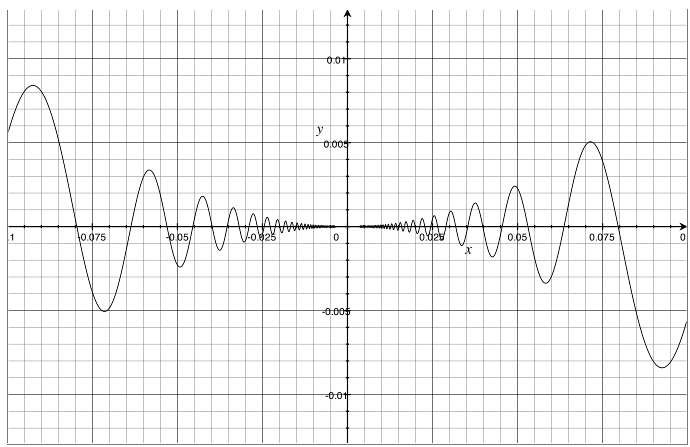

Figure 1: $y = x^2\sin(1/x)$

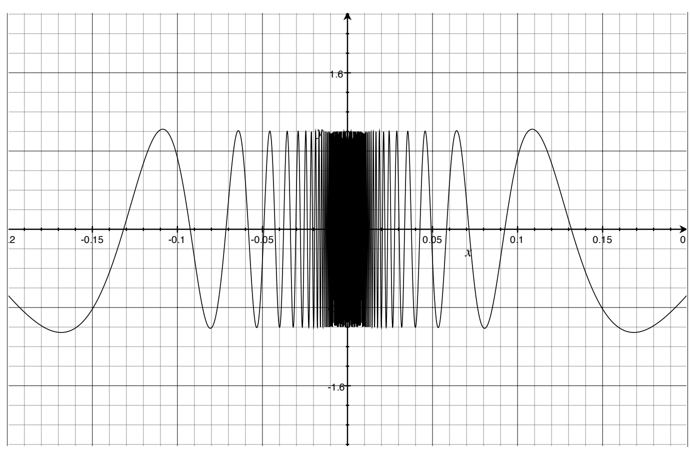

Figure 2: $y = 2x\sin(1/x) - \cos(1/x)$