Name: *Answer Key* 12/19/2017

Math 203 Calculus III Final Exam

Part I: Answer all 7 questions (10 points each)

1. (a) Give an equation for the plane through the following three points.

$$A = (1, 1, 1) \quad B = (-3, 2, 5) \quad C = (4, -1, 6)$$

NORMAL VECTOR $\perp$ TO $\vec{AB}$ & $\vec{AC}$ $\Rightarrow$ $\parallel$ TO $\vec{AB} \times \vec{AC}$

$\vec{AB} = \langle -4, 1, 4 \rangle$, $\vec{AC} = \langle 3, -2, 5 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 1 & 4 \\ 3 & -2 & 5 \end{vmatrix} = \langle 5+8, 12+20, 8-3 \rangle = \langle 13, 32, 5 \rangle$$

$\therefore$ EQ: $13x + 32y + 5z = c$, $(1,1,1)$ ON PLANE $\Rightarrow$ $13(1) + 32(1) + 5(1) = c$

$50 = c$

$\therefore$ $\boxed{13x + 32y + 5z = 50}$

(b) The line l has the following parametric equations. DIRECTION VECTOR $\vec{v} = \langle -2, -5, -3 \rangle$

$$x = 4 - 2t, \quad y = 3 - 5t, \quad z = -3 - 3t \qquad (l)$$

Determine whether each of the following planes is parallel to l, perpendicular to l, or neither .

i. $x + 5y - 4z = -9$ NEITHER $\because$ $\vec{n} \neq c\vec{v}$ AND $\vec{n} \cdot \vec{v} = -15 \neq 0$
$\vec{n} = \langle 1, 5, -4 \rangle$

ii. $4x + 10y + 6z = 16$ PERPINDICULAR $\because$ $\vec{n} = -2\vec{v}$ ($\vec{n} \parallel \vec{v} \Rightarrow \vec{v} \perp$ PLANE)
$\vec{n} = \langle 4, 10, 6 \rangle$

iii. $-25x - 11y + 35z = 23$ PARALLEL $\because$ $\vec{n} \cdot \vec{v} = 0$ ($\vec{n} \perp \vec{v} \Rightarrow \vec{v} \parallel$ PLANE)
$\vec{n} = \langle -25, -11, 35 \rangle$

(c) Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane

$$x - y + 3z = 7.$$

$\perp$ TO PLANE $\Rightarrow$ $\parallel$ TO $\vec{n} = \langle 1, -1, 3 \rangle$

$\vec{r}(t) = \langle 2, 4, 6 \rangle + t \langle 1, -1, 3 \rangle$ i.e.

$$\boxed{\begin{aligned} x &= 2 + t \\ y &= 4 - t \\ z &= 6 + 3t \end{aligned}}$$

2. (a) Give a unit tangent vector to the following parametric curve at the point $(4, 2, 7)$.

$$\mathbf{r}(t) = \langle t^3 + 3t, t^2 + 1, 3t + 4 \rangle$$

$\vec{r}\,'(t) = \langle 3t^2 + 3,\ 2t,\ 3 \rangle$

NOTE THAT $\vec{r}(1) = \langle 4, 2, 7 \rangle$

$\vec{r}\,'(1) = \langle 6, 2, 3 \rangle \longrightarrow$ UNIT LENGTH : $\dfrac{\vec{r}\,'(1)}{|\vec{r}\,'(1)|} = \dfrac{\langle 6, 2, 3 \rangle}{\sqrt{6^2 + 2^2 + 3^2}}$

$= \boxed{\frac{1}{7}\langle 6, 2, 3 \rangle}$

(b) Give an equation for the tangent plane to the following surface at the point $(0, 0, 3)$.

$$z = \frac{2x + 3}{4y + 1} = f(x, y)$$

$f_x(x, y) = \dfrac{2}{4y + 1}$, $f_y(x, y) = \dfrac{-4(2x + 3)}{(4y + 1)^2}$

$f_x(0, 0) = 2$, $f_y(0, 0) = -12$

TANGENT PLANE : $z = f(0,0) + f_x(0,0)(x - 0) + f_y(0,0)(y - 0)$

$\boxed{z = 3 + 2x - 12y}$

3. For each of the following, evaluate the limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$

ALONG X-AXIS : $\displaystyle\lim_{x\to 0} \frac{0}{y^4} = 0$

ALONG Y-AXIS : $\displaystyle\lim_{y\to 0} \frac{5y^4}{y^4} = 5$

NOT EQUAL $\Rightarrow$ LIMIT $\boxed{\text{D.N.E.}}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

POLAR COORDINATES :

SET $x = r\cos\theta$, $y = r\sin\theta$

AND LET $r \to 0$

$$= \lim_{r\to 0} \frac{r^2 \sin\theta\cos\theta}{r} = \lim_{r\to 0} r\sin\theta\cos\theta = \boxed{0}$$

NOTE THAT THE LIMIT DOES NOT DEPEND ON θ.

4. Suppose that over a certain region of space the electrical potential V is given by

$$V(x, y, z) = 5x^2 - 3xy + xyz.$$

(a) Find the rate of change of V at the point $P = (3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

i.e. FIND $D_{\vec{u}} V(3,4,5) = \nabla V(3,4,5) \cdot \vec{u}$

WHERE $\vec{u} = \dfrac{\langle 1, 1, -1 \rangle}{\sqrt{1^2 + 1^2 + (-1)^2}} = \dfrac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle$

AND $\nabla V(x,y,z) = \langle 10x - 3y + yz, -3x + xz, xy \rangle$

$\therefore \nabla V(3,4,5) \cdot \frac{1}{\sqrt{3}} \langle 1,1,-1 \rangle = \langle 38, 6, 12 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle = \boxed{\dfrac{32}{\sqrt{3}}}$

(b) In which direction does V change most rapidly at P?

$\nabla V(3,4,5) = \boxed{\langle 38, 6, 12 \rangle \text{ OR ANY POSITIVE MULTIPLE OF THIS VECTOR}}$

(c) Now suppose that

$$x = s + t, \quad y = 2s - 3t, \quad z = st^2.$$

Find $\dfrac{\partial V}{\partial t}$ at the point $(s, t) = (1, 1)$.

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial V}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial V}{\partial z} \cdot \frac{\partial z}{\partial t} = \nabla V \cdot \left\langle \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t} \right\rangle$$

WHEN $s = 1, t = 1, x = 2, y = -1, z = 1$

$$\frac{\partial V}{\partial t} = \langle 20 + 3 - 1, -6 + 2, -2 \rangle \cdot \langle 1, -3, 2 \rangle$$

$$= \langle 22, -4, -2 \rangle \cdot \langle 1, -3, 2 \rangle = 22 + 12 - 4 = \boxed{30}$$

5. Find all critical points for the function

$$f(x,y) = y^2 - 2y\cos x \quad , \quad 0 \le x \le 2\pi$$

and classify each one as a local maximum, local minimum, or saddle point.

$f_x = 2y \sin x = 0$, $f_y = 2y - 2\cos x = 0$

$y = \cos x$

$y = 0$ $\quad$ $x = 0, \pi, 2\pi$

IF $y = 0$ THEN $\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\boxed{(\tfrac{\pi}{2}, 0), (\tfrac{3\pi}{2}, 0) \quad \text{SADDLE POINTS}}$

IF $x = 0$ OR 2π THEN $y = \cos(0) = \cos(2\pi) = 1$

$\boxed{(0, 1), (2\pi, 1) \quad \text{LOCAL MIN}}$

IF $x = \pi$ THEN $y = \cos\pi = -1$

$\boxed{(\pi, -1) \quad \text{LOCAL MIN}}$

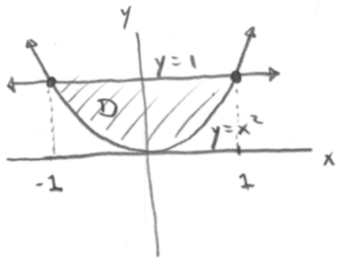
$f_{xx} = 2y\cos x$, $f_{yy} = 2$, $f_{xy} = f_{yx} = 2\sin x$

$\therefore \; D = f_{xx}f_{yy} - f_{xy}^2 = 4y\cos x - 4\sin^2 x$

CRITICAL POINT	D	f_{xx}	CLASSIFICATION
$(\frac{\pi}{2}, 0)$	-4	N/A	SADDLE PNT
$(\frac{3\pi}{2}, 0)$	-4	N/A	SADDLE PNT
$(0, 1)$	4	2	LOCAL MIN
$(2\pi, 1)$	4	2	LOCAL MIN
$(\pi, -1)$	4	2	LOCAL MIN

6. (a) Find the volume of the solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $\underbrace{y + z = 1}_{z = 1 - y}$.

PLANES $z = 0$ & $y + z = 1$ INTERSECT : $y = 1$



CURVES INTERSECT :

$x^2 = 1$

$x = \pm 1$

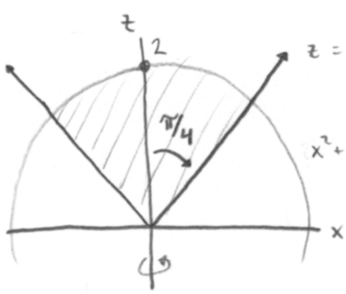
$$\text{VOLUME} = \iint_D (1 - y)\, dA$$

$$= \int_{-1}^{1} \int_{x^2}^{1} (1 - y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[y - \tfrac{1}{2} y^2 \right]_{x^2}^{1} dx = \int_{-1}^{1} \tfrac{1}{2} - \left(x^2 - \tfrac{1}{2} x^4\right) dx$$

$$= \tfrac{1}{2} x - \tfrac{1}{3} x^3 + \tfrac{1}{10} x^5 \Big|_{-1}^{1} = \boxed{\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{10}\right) - \left(-\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{10}\right) = \tfrac{8}{15}}$$

(b) Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 4$.



$z = \sqrt{x^2} = |x|$

$x^2 + y^2 + z^2 = 4$

XZ-PLANE $(y = 0)$

SPHERICAL COORDINATES:

$0 \le \rho \le 2$

$0 \le \phi \le \pi/4$

$0 \le \theta \le 2\pi$

$$\text{VOLUME} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi\, d\phi \int_0^{2} \rho^2\, d\rho = 2\pi \left[-\cos\phi \right]_0^{\pi/4} \tfrac{1}{3} \rho^3 \Big|_0^2$$

$$= 2\pi \left(1 - \tfrac{\sqrt{2}}{2}\right) \tfrac{8}{3} = \boxed{\tfrac{16\pi}{3}\left(1 - \tfrac{\sqrt{2}}{2}\right)}$$

$$= \frac{32\pi - 16\pi\sqrt{2}}{6}$$

7. For each of the following series, state whether it is absolutely convergent, conditionally convergent, or divergent. Name a test which supports each conclusion, and show the work to apply the test.

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

INTEGRAL TEST : $\int_{2}^{\infty} \frac{1}{x \ln x} \, dx$ LET $u = \ln x$, $du = \frac{1}{x} dx$

$$\rightsquigarrow \int_{\ln 2}^{\infty} \frac{1}{u} \, dx = \ln|u| \Big|_{\ln 2}^{\infty} = \infty$$

$\therefore$ $\boxed{\text{DIVERGENT}}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$

ALT SERIES TEST : $\lim_{n \to 0} \left| \frac{(-1)^n}{5n+1} \right| = 0$ $\therefore$ CONVERGENT

ABSOLUTELY ?

$$\sum_{n=0}^{\infty} \frac{1}{5n+1} \geq \sum_{n=0}^{\infty} \frac{1}{5n+5} = \sum_{n=0}^{\infty} \frac{1}{5(n+1)} = \sum_{n=1}^{\infty} \frac{1}{5n} = \frac{1}{5} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}$$

HARMONIC SERIES DIVERGES.

$\therefore$ BY COMPARISON TEST, THE SERIES DOES <u>NOT</u> CONVERGE ABSOLUTELY.

$\therefore$ $\boxed{\text{CONDITIONALLY CONVERGENT}}$

(c) $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1} \right)^n$

ROOT TEST : $\lim_{n \to \infty} \sqrt[n]{\left(\frac{n^2+1}{2n^2+1} \right)^n}$

$$= \lim_{n \to \infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1$$

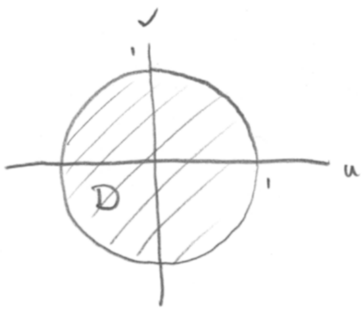
$\therefore$ $\boxed{\text{ABSOLUTELY CONVERGENT}}$

(ALL TERMS ARE ALREADY POSITIVE)

Part II: Answer 3 out of 5 questions (10 points each)

8. Find the area of the following parametric surface.

$$\mathbf{r}(u, v) = \langle u\cos v, u\sin v, v\rangle, \quad u^2 + v^2 \leq 1$$



$$\text{SURFACE AREA} = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

$$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$$

$$\vec{r}_v = \langle -u\sin v, u\cos v, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u\cos^2 v + u\sin^2 v \rangle$$

$$= \langle \sin v, -\cos v, u \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$$

$$\text{SURFACE AREA} = \iint_D \sqrt{1+u^2} \, dA = \int_0^{2\pi} \left[\int_0^1 \sqrt{1 + r^2\cos^2\theta} \; r\,dr \right] d\theta$$

LET $w = 1 + r^2\cos^2\theta$

$dw = 2r\cos^2\theta \, dr \Rightarrow \frac{1}{2}\sec^2\theta \, dw = r\,dr$

$$= \frac{1}{2}\int_0^{2\pi} \sec^2\theta \left[\int_1^{1+\cos^2\theta} \sqrt{w} \, dw \right] d\theta =$$

$$\frac{1}{2\sqrt{w}} \Bigg|_1^{1+\cos^2\theta} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\cos^2\theta}}\right.$$

9. (a) In spherical coordinates, the cone $9z^2 = x^2 + y^2$ has equation $\phi = c$. Find c.

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$$9\rho^2\cos^2\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$$

$$9\rho^2\cos^2\phi = \rho^2\sin^2\phi$$

$$9 = \tan^2\phi$$

$$\pm 3 = \tan\phi$$

$$\boxed{\phi = \pm \tan^{-1}(3)}$$

BUT $\phi \geq 0$

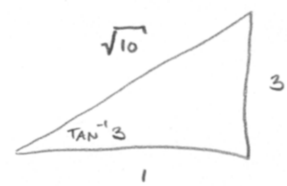
SO WE TAKE $c = \tan^{-1}(3)$

(b) Find

$$\iiint_R (x^2 + y^2 + z^2)^{3/2}\, dV,$$

where R is the region inside the sphere $x^2 + y^2 + z^2 = 3$ and inside the cone $z = \dfrac{\sqrt{x^2+y^2}}{3}$.

$$\int_0^{2\pi}\int_0^{\tan^{-1}(3)}\int_0^{\sqrt{3}} (\rho^2)^{3/2}\, \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$



$$= \int_0^{2\pi} d\theta \int_0^{\tan^{-1}3} \sin\phi\, d\phi \int_0^{\sqrt{3}} \rho^5\, d\rho$$

$$= 2\pi \Big[-\cos\phi\Big]_0^{\tan^{-1}3} \; \frac{1}{6}\rho^6 \Big|_0^{\sqrt{3}}$$

$$= 2\pi\Big[1 - \cos(\tan^{-1}3)\Big]\frac{3^3}{6} = \boxed{9\pi\left(1 - \frac{1}{\sqrt{10}}\right)}$$

10. Let

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Show that $f_x(0,0)$ and $f_y(0,0)$ both exist. Note that you must use the definition of partial derivatives as limits.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{\frac{0}{h^2} - 0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \frac{\frac{0}{h^2} - 0}{h} = 0$$

BOTH EXIST. ✓

(b) Explain why f is not differentiable at $(0,0)$.

REASON 1: $\boxed{f \text{ IS NOT CONTINUOUS AT } (0,0)}$

ALONG X-AXIS: $\lim_{x\to 0} \frac{0}{x^2} = 0$ ALONG $y=x$: $\lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

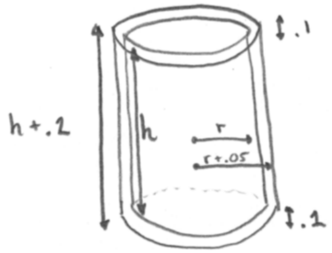
REASON 2:

$$f_x(x,y) = \begin{cases} \dfrac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \dfrac{y^3 - x^2y}{(x^2+y^2)^2} & \text{IF } (x,y) \neq 0 \\ 0 & \text{IF } (x,y) = 0 \end{cases}$$

$\boxed{f_x \text{ IS NOT CONTINUOUS AT } (0,0)}$

ALONG X-AXIS: $\lim_{x\to 0} \frac{0}{x^4} = 0$ ALONG Y-AXIS: $\lim_{y\to 0} \frac{y^3}{y^4} = \infty$

11. (a) Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.



$$V = \pi r^2 h$$

$$dV = 2\pi r h\, dr + \pi r^2 dh$$

$$r = 2 \qquad h = 10$$

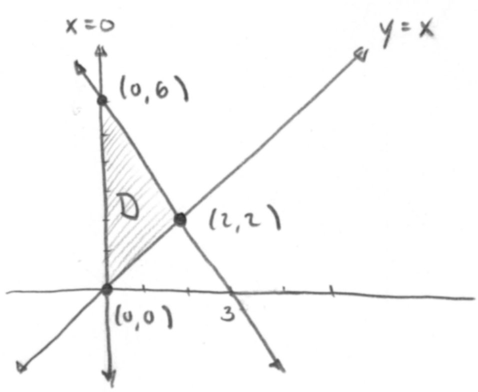
$$dr = .05 \qquad dh = .2$$

$$dV = 2\pi(2)(10)(.05) + \pi(2)^2(.2)$$

$$= 2\pi + .8\pi = \boxed{2.8\pi \text{ cm}^3}$$

(b) Find the mass of the lamina that occupies the triangular region enclosed by the following lines and has density $\delta(x, y) = x^2$.

$$x = 0, \quad \underbrace{y = x, \quad 2x + y = 6}$$



INTERSECT : $x = 6 - 2x$

$3x = 6$

$x = 2, \quad y = 2$

$$\text{MASS} = \iint_D \delta(x,y)\, dA$$

$$= \int_0^2 \underbrace{\int_x^{6-2x} x^2\, dy}\, dx = \int_0^2 6x^2 - 3x^3\, dx = 2x^3 - \frac{3}{4}x^4 \Big|_0^2 = 16 - 12 = \boxed{4}$$

$$x^2 y \Big|_{y=x}^{y=6-2x} = x^2\left[(6-2x) - x\right] = x^2(6-3x) = 6x^2 - 3x^3$$

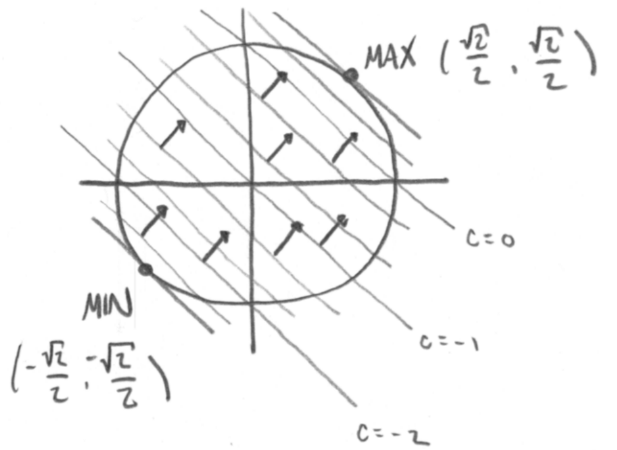
12. For each of the following functions, find the maximum and minimum values of the function on the circular disk $x^2 + y^2 \leq 1$. Do this by first sketching the level curves and gradients.

(a) $f(x, y) = x + y + 1$

LEVEL CURVES : $f(x,y) = x + y + 1 = c$

$y = -x + 1 + c$

$\nabla f = \langle 1, 1 \rangle$



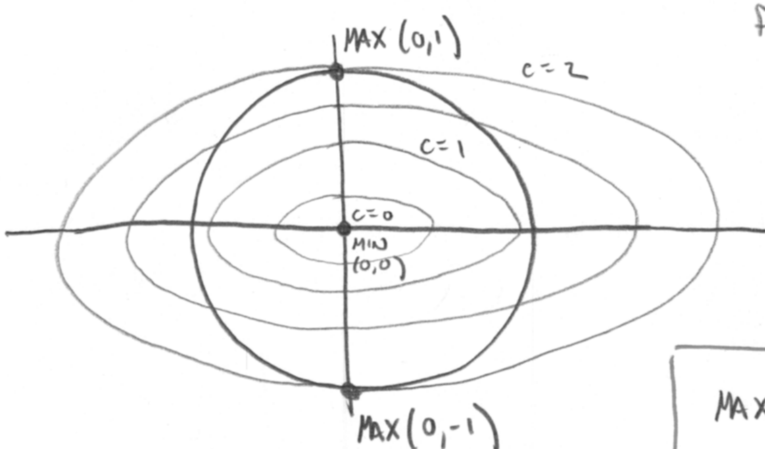
$$\boxed{\begin{aligned} &\text{MAX } f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2} + 1 \\ &\text{MIN } f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 1 - \sqrt{2} \end{aligned}}$$

(b) $f(x, y) = x^2 + 2y^2$

LEVEL CURVES :

$f(x,y) = x^2 + 2y^2 = c$

$\frac{x^2}{2} + y^2 = \frac{c}{2}$ (ELLIPSE)



$$\boxed{\begin{aligned} &\text{MAX } f(0,1) = 2 \\ &\text{MIN } f(0,0) = 0 \end{aligned}}$$