Name: *ANSWER KEY* 10/8/2017

Math 203 Calculus III Practice Exam 1

1. Find equations of the spheres with center $(2,3,6)$ that touch

   (a) the xy-plane; DISTANCE TO XY-PLANE = ABS VAL OF Z COORD.

   $$\boxed{(x-2)^2+(y-3)^2+(z-6)^2=6^2}$$

   (b) the yz-plane; DISTANCE TO YZ PLANE = ABS VAL OF X COORD.

   $$\boxed{(x-2)^2+(y-3)+(z-6)^2=2^2}$$

   (c) the xz-plane. DISTANCE TO XZ-PLANE = ABS VAL OF Y COORD

   $$\boxed{(x-2)^2+(y-3)^2+(z-6)^2=3^2}$$

2. (a) Find the acute angle between the two vectors $\mathbf{a}=\langle 4,0,2\rangle$ and $\mathbf{b}=\langle 2,-1,0\rangle$.

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{4\cdot 2+0(-1)+2(0)}{\sqrt{4^2+0^2+2^2}\sqrt{2^2+(-1)^2+0^2}}=\frac{8}{\sqrt{20}\sqrt{5}}=\frac{4}{5}$$

$$\boxed{\theta=\cos^{-1}\left(\frac{4}{5}\right)}$$

   (b) Find the acute angle between the following lines.

$$2x-y=3 \qquad 3x+y=7$$

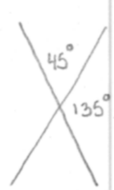
$y=2x-3$ SLOPE 2 $\qquad$ $y=-3x+7$ SLOPE -3

2 VECTORS PARALLEL TO THESE LINES, RESPECTIVELY:

SLOPE 2 : $\langle 1,2\rangle$ $\qquad$ SLOPE -3 : $\langle 1,-3\rangle$ $\qquad$ (NOW IT'S JUST LIKE QUESTION 2 WITH THESE VECTORS)

$$\cos\theta=\frac{\langle 1,2\rangle\cdot\langle 1,-3\rangle}{|\langle 1,2\rangle||\langle 1,-3\rangle|}=\frac{1-6}{\sqrt{5}\sqrt{10}}=\frac{-5}{\sqrt{50}}=-\frac{1}{\sqrt{2}}$$

$$\theta=\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)=135^\circ$$



$\therefore$ ACUTE ANGLE $=\boxed{45^\circ}$

3. Find the scalar and vector projections of $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ onto $\mathbf{a} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$, i.e. $\text{comp}_{\mathbf{a}}\,\mathbf{b}$ and $\text{proj}_{\mathbf{a}}\,\mathbf{b}$.

$$\text{comp}_{\vec{a}}\,\vec{b} = \vec{b}\cdot\frac{\vec{a}}{|\vec{a}|} = \frac{\langle 1,2,3\rangle\cdot\langle 3,6,-2\rangle}{\sqrt{3^2+6^2+(-2)^2}} = \boxed{\frac{9}{7}}$$

$$\text{proj}_{\vec{a}}\,\vec{b} = \text{comp}_{\vec{a}}\,\vec{b}\left(\frac{\vec{a}}{|\vec{a}|}\right) = \frac{9}{7}\left(\frac{\langle 3,6,-2\rangle}{7}\right)$$

$$= \boxed{\frac{9}{49}\langle 3,6,-2\rangle}$$

4. Find two unit vectors orthogonal to both $\mathbf{u} = \langle 3,2,1\rangle$ and $\mathbf{v} = \langle 1,1,0\rangle$.

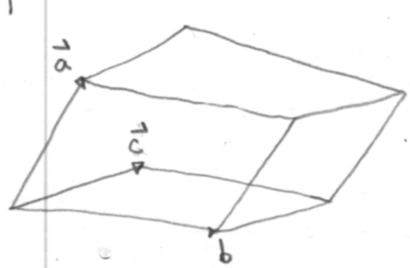
$$\perp:\quad \pm\,\vec{u}\times\vec{v} = \pm\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \pm\langle -1, 1, 1\rangle$$

NOW MAKE THEM <u>UNIT</u> VECTORS : $\pm\dfrac{\langle -1,1,1\rangle}{|\langle -1,1,1\rangle|} = \boxed{\pm\frac{1}{\sqrt{3}}\langle -1,1,1\rangle}$

5. Find the volume of the parallelepiped ("parallelo-box") determined by the vectors $\mathbf{a} = \langle 1,2,3\rangle$, $\mathbf{b} = \langle 1,1,2\rangle$, $\mathbf{c} = \langle 2,1,4\rangle$.

$$\text{Vol} = |\vec{a}\cdot(\vec{b}\times\vec{c})| = \left|\vec{a}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix}\right| = |\langle 1,2,3\rangle\cdot\langle 2,0,-1\rangle|$$

$$= |2-3| = \boxed{1}$$



6. Use the scalar triple product to determine whether the points $A(1,3,2,)$ $B(3,-1,6)$, $C(5,2,0)$, and $D(3,6,-4)$ lie in the same plane.

3 VECTORS : $\vec{AB} = \langle 2,-4,4\rangle$, $\vec{AC} = \langle 4,-1,-2\rangle$, $\vec{AD} = \langle 2,3,-6\rangle$

THE 4 PNTS ARE COLINEAR IF THESE 3 VECTORS DESCRIBE A PARALLELOPIPED WITH 0 VOLUME.

$$\text{Vol} = |\vec{AB}\cdot(\vec{AC}\times\vec{AD})| = \left|\vec{AB}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix}\right| = \langle 2,-4,4\rangle\cdot\langle 12,20,14\rangle$$

$$= 24 - 80 + 56 = 0$$

$\boxed{\text{YES, COPLANAR}}$

7. Find a vector equation and parametric equations for the line through the point $(3,-4,6)$ and

(a) perpendicular to the plane $x+3y-z=11$.

i.e. PARALLEL TO NORMAL VECTOR $\langle 1,3,-1\rangle$

$\therefore \boxed{\vec{r}(t) = \langle 3,-4,6\rangle + t\langle 1,3,-1\rangle}$ OR $\boxed{\begin{aligned} x &= 3+t \\ y &= -4+3t \\ z &= 6-t \end{aligned}}$

(b) parallel to the vector $\mathbf{v} = \langle 1,3,-1\rangle$.

SAME AS PART (a)

(c) Briefly explain why the same answer works for both parts (a) and (b).

PERPINDICULAR TO THE PLANE MEANS PARALLEL TO THE PLANE'S NORMAL VECTOR.

PARALLEL TO THE PLANE MEANS PERPINDICULAR TO THE PLANE'S NORMAL VECTOR.

8. Find a vector equation and parametric equations for the line *segment* from $(7,3,-1)$ to $(-2,4,5)$.

$\boxed{\vec{r}(t) = (1-t)\langle 7,3,-1\rangle + t\langle -2,4,5\rangle, \quad 0\le t\le 1}$

OR $\boxed{\begin{aligned} x &= 7-9t \\ y &= 3+t, \quad 0\le t\le 1 \\ z &= -1+6t \end{aligned}}$

IMPORTANT! OTHERWISE IT IS NOT A LINE SEGMENT

9. Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x+y+z=2$ and perpendicular to the line

$$x = 1+t, \qquad y = 1-t, \qquad z = 2t.$$

DIRECTION VECTOR

NORMAL VECTOR

$\vec{r}(t) = \langle 0, 1, 2 \rangle + t\vec{v}$, WHERE $\vec{v}$ IS PERPINDULAR TO $\langle 1, -1, 2 \rangle$

AND PERPINDICULAR TO $\langle 1, 1, 1 \rangle$

TAKE $\vec{v} = \langle 1, -1, 2 \rangle \times \langle 1, 1, 1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \langle -3, 1, 2 \rangle$

$\therefore\ \vec{r}(t) = \langle 0, 1, 2 \rangle + t\langle -3, 1, 2 \rangle$ OR $\boxed{\begin{aligned} x &= -3t \\ y &= 1+t \\ z &= 2+2t \end{aligned}}$

10. Find parametric equations for the tangent line to the curve $\mathbf{r}(t) = \langle t^3+3t, t^2+1, 3t+4 \rangle$ at the point $(4, 2, 7)$.

Note: $\vec{r}(1) = \langle 4, 2, 7 \rangle$

$\therefore$ TANGENT LINE $\ell(t) = \vec{r}(1) + t\vec{r}\,'(1)$

$\vec{r}\,'(t) = \langle 3t^2+3, 2t, 3 \rangle$

$\vec{r}\,'(1) = \langle 6, 2, 3 \rangle$

$\therefore\ \ell(t) = \langle 4, 2, 7 \rangle + t\langle 6, 2, 3 \rangle$

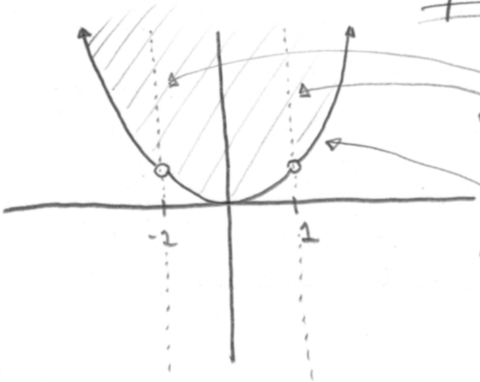
i.e. $\boxed{\begin{aligned} x &= 4+6t \\ y &= 2+2t \\ z &= 7+3t \end{aligned}}$

11. Sketch the domain of the function $f(x, y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$.

WE MUST HAVE ① $y - x^2 \geq 0$ AND ② $1 - x^2 \neq 0$

$y \geq x^2$

$x \neq \pm 1$



POINTS ON THESE LINES REMOVED

POINTS ON PARABOLA INCLUDED (EXCEPT $(\pm 1, 1)$)

12. (a) Show that the following limit does not exist.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$$

ALONG THE X-AXIS $(y = z = 0)$: $\lim_{x \to 0} \frac{0}{x^2} = 0$

IN THE xy-PLANE $(z = 0)$ ALONG THE LINE $y = x$: $\lim_{x \to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

SINCE THESE 2 LIMITS ARE DIFFERENT, THE LIMIT DOES NOT EXIST.

(b) Show that the following limit does exist, and find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

USING POLAR COORD, WE HAVE $x = r\cos\theta$, $y = r\sin\theta$ AND $x^2 + y^2 = r^2$.

AS $(x,y) \to (0,0)$, $r \to 0$

$$\therefore \lim_{r \to 0} \frac{r^2}{\sqrt{r^2+1} - 1} = \lim_{r \to 0} \frac{r^2}{\sqrt{r^2+1} - 1} \cdot \frac{\sqrt{r^2+1} + 1}{\sqrt{r^2+1} + 1}$$

$$= \lim_{r \to 0} \frac{\cancel{r^2}\left(\sqrt{r^2+1} + 1\right)}{\cancel{r^2}} = \boxed{2}$$