Name: * ANSWER KEY *

Math 203 Calculus III

11/16/2017

Practice Exam 2

1. Let $f(x, y) = 1 + x\ln(xy - 5)$. Explain why f is differentiable at the point $(2, 3)$. Then find the linearization $L(x, y)$ of the function at that point.

f IS DIFFERENTIABLE AT $(2,3)$ IF f_x & f_y ARE BOTH CONTINUOUS IN A NEIGHBORHOOD OF $(2,3)$.

$$f_x = \ln(xy-5) + \frac{xy}{xy-5}, \quad f_y = \frac{x^2}{xy-5}$$

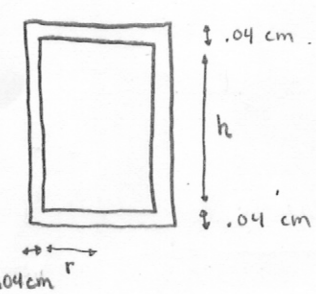
← CONTINUOUS ON THEIR DOMAIN ✓ (WHICH INCLUDES $(2,3)$)

$$L(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3)$$

$$\boxed{L(x,y) = 1 + 6(x-2) + 4(y-3)}$$

2. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

SIDE VIEW



$dh = \Delta h = .08$ cm

$dr = \Delta r = .04$ cm

$$V(r,h) = \pi r^2 h$$

$$dV = V_r\,dr + V_h\,dh$$

$$dV = 2\pi r h\,dr + \pi r^2\,dh$$

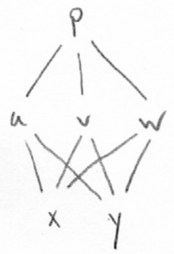
$$\therefore \left. dV \right|_{r=4,\ dr=.04,\ h=12,\ dh=.08} = 2\pi(4)(12)(.04) + \pi(4)^2(.08)$$

$$= 2.92\pi + 1.28\pi = \boxed{4.2\pi \text{ cm}^3}$$

3. Let

$$P = \sqrt{u^2 + v^2 + w^2}, \quad u = xe^y, \quad v = ye^x, \quad w = e^{xy}.$$

Use the Chain Rule for several variables to find $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ when $x = 0$ and $y = 2$.



$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{1}{\sqrt{u^2 + v^2 + w^2}}\left(u \cdot e^y + v \cdot ye^x + w \cdot ye^{xy}\right)$$

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{1}{\sqrt{u^2 + v^2 + w^2}}\left(u \cdot xe^y + v \cdot e^x + w \cdot xe^{xy}\right)$$

WHEN $x = 0$ & $y = 2$, $u = 0$, $v = 2$, $w = 1$

$$\therefore \boxed{\left.\frac{\partial P}{\partial x}\right|_{\substack{x=0\\y=2}} = \frac{1}{\sqrt{5}}(4 + 2) \quad \& \quad \left.\frac{\partial P}{\partial y}\right|_{\substack{x=0\\y=2}} = \frac{1}{\sqrt{5}}(2)}$$

4. Find the maximum rate of change of $f(x, y, z) = \tan^{-1}(xyz)$ at the point $(1, 2, 1)$ and the direction in which it occurs.

MAX $D_{\vec{u}} f(x,y,z) = \|\nabla f(x,y,z)\|$

DIRECTION IN WHICH IT OCCURS IS $\nabla f(x,y,z)$

$$\nabla f = \frac{1}{1 + (xyz)^2}\langle yz, xz, xy\rangle$$

MAX $D_{\vec{u}} f(1,2,1) = \|\nabla f(1,2,1)\| = \frac{1}{\sqrt{5}}\|\langle 2, 1, 2\rangle\| = \frac{\sqrt{2^2 + 1^2 + 2^2}}{\sqrt{5}} = \boxed{\frac{3}{\sqrt{5}}}$

OCCURS IN DIRECTION PARALLEL TO $\nabla f(1,2,1) = \boxed{\frac{1}{\sqrt{5}}\langle 2, 1, 2\rangle}$

5. After drifting, the height h in inches of the snow at point (x, y) in a parking lot is

$$h(x, y) = 4 + x^2 - \ln(y^2 + 1).$$

(a) Find the rate at which the height of the snow at the point $(3, 1)$ changes per unit distance in the direction toward the point $(4, 0)$.

FROM $(3,1)$ TO $(4,0)$ IS $\vec{v} = \langle 1, -1 \rangle$

UNIT VECTOR $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{2}} \langle 1, -1 \rangle$

FIND $D_{\vec{u}} f(3,1) = \nabla f(3,1) \cdot \vec{u}$

SINCE $\nabla f = \left\langle 2x, \dfrac{-2y}{y^2+1} \right\rangle \longrightarrow \nabla f(3,1) = \langle 6, -1 \rangle$,

$$D_{\vec{u}} f(3,1) = \langle 6, -1 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, -1 \rangle = \frac{1}{\sqrt{2}}(6+1) = \boxed{\frac{7}{\sqrt{2}}}$$

(b) Suppose a person is walking in the parking lot and their position at time t is given by the following parametric equations.

$$x = 2t, \quad y = \sin t, \quad 0 \le t$$

Find the rate at which the height of the snow the person is walking through is changing per unit time when the person is at the point $(\pi, 1)$.

NOTE: THIS POINT CORRESPONDS TO $t = \frac{\pi}{2}$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (2x)(2) + \left(\frac{-2y}{y^2+1}\right)(\cos t)$$

$$\left.\frac{df}{dt}\right|_{\substack{x=\pi \\ y=1 \\ t=\frac{\pi}{2}}} = \boxed{4\pi}$$

6. Find the absolute maximum and absolute minimum value of

$$f(x, y) = x^4 + y^4 - 4xy + 2, \quad 0 \le x \le 3, \quad 0 \le y \le 2.$$

$f_x = 4x^3 - 4y = 0$ $\qquad$ $f_y = 4y^3 - 4x = 0$

$\Rightarrow y = x^3 \longrightarrow 4(x^3)^3 - 4x = 0$

$4x(x^8 - 1) = 0 \longrightarrow 4x(x^4+1)(x^2+1)(x+1)(x-1) = 0$

$x = 0, \pm 1$

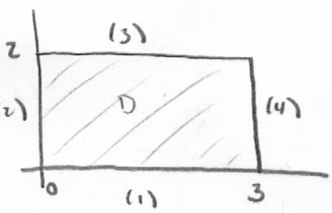
IF $x = 0$, $y = 0$

IF $x = 1$, $y = 1$

IF $x = -1$, $y = -1$

$\therefore$ CRITICAL POINTS : $(0,0)$, $(1,1)$, $(-1,-1)$

$f(0,0) = 2$, $f(1,1) = 0$, $f(-1,-1) = 0$



BOUNDARY OF D HAS 4 COMPONENTS.

FIND MAX/MIN OF f ON EACH COMPONENT.

$$\boxed{\begin{array}{l}\text{ABS. MAX. VALUE } 83 \\ \text{ABS. MIN. VALUE } 0\end{array}}$$

(1) $y = 0 \rightarrow f(x) = x^4 + 2$, MIN 2, MAX 83
$(0 \le x \le 3)$

(2) $x = 0 \rightarrow f(y) = y^4 + 2$, MIN 2, MAX 18
$(0 \le y \le 2)$

(3) $y = 2 \rightarrow f(x) = x^4 - 8x + 18 \rightarrow f'(x) = 4x^3 - 8 = 0 \rightarrow x = \sqrt[3]{2}$ CRITICAL PNT.

$f(0) = 18$, $f(\sqrt[3]{2}) = 2\sqrt[3]{2} - 8\sqrt[3]{2} + 18$ (MIN), $f(3) = 75$

(4) $x = 3 \rightarrow f(y) = y^4 - 12y + 83 \rightarrow f'(y) = 4y^3 - 12 = 0 \rightarrow y = \sqrt[3]{3}$ CRITICAL PNT.

$f(0) = 83$, $f(\sqrt[3]{3}) = 3\sqrt[3]{3} - 12\sqrt[3]{3} + 83$ (MIN), $f(2) = 75$

7. Find all local maxima, local minima, and saddle points for

$$f(x,y) = 2x^4 - x^2 + 3y^2.$$

CRITICAL POINTS: $f_x = 8x^3 - 2x = 0$ $\qquad$ $f_y = 6y = 0$

$2x(4x^2 - 1) = 0$ $\qquad$ $y = 0$

$x = 0$ $\qquad$ $x = \pm \frac{1}{2}$

3 CRITICAL PNTS: $(0,0)$, $(\frac{1}{2}, 0)$, $(-\frac{1}{2}, 0)$

$f_{xx} = 24x^2 - 2$ $\qquad$ $f_{yy} = 6$ $\qquad$ $f_{xy} = f_{yx} = 0$

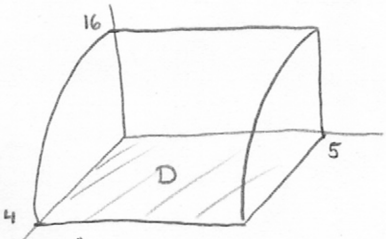
$D = f_{xx} f_{yy} - f_{xy}^2 = 6(24x^2 - 2)$

$(0,0)$: $D = -12 < 0 \Rightarrow$ $\boxed{(0,0) \text{ IS SADDLE POINT}}$

$(\frac{1}{2}, 0)$: $D = 24 > 0$ AND $f_{xx} = 4 > 0 \Rightarrow$ $\boxed{(\frac{1}{2}, 0) \text{ IS LOCAL MIN}}$

$(-\frac{1}{2}, 0)$: $D = 24 > 0$ AND $f_{xx} = 4 > 0 \Rightarrow$ $\boxed{(-\frac{1}{2}, 0) \text{ IS LOCAL MIN}}$

8. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.



$16 - x^2 = 0$, $x \geq 0$

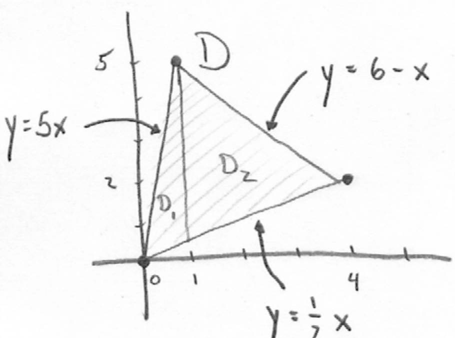
$\Rightarrow x = 4$

$$V = \iint_D (16 - x^2)\, dA = \int_0^5 \int_0^4 (16 - x^2)\, dx\, dy$$

$$= \int_0^5 dy \int_0^4 (16 - x^2)\, dx = 5\left[16x - \frac{1}{3}x^3\right]_{x=0}^{x=4}$$

$$= 5\left[64 - \frac{64}{3}\right] = 5 \cdot \frac{128}{3} = \boxed{\frac{640}{3}}$$

9. Find the volume of the solid that lies below the surface $z = x^2 y$ and above the triangular region in the xy-plane with vertices $(0, 0)$, $(4, 2)$ and $(1, 5)$.



(UNION OF 2 TYPE I REGIONS)

$$V = \iint_D x^2 y\, dA = \iint_{D_1} x^2 y\, dA + \iint_{D_2} x^2 y\, dA$$

$$= \int_0^1 \int_{\frac{1}{2}x}^{5x} x^2 y\, dy\, dx + \int_1^4 \int_{\frac{1}{2}x}^{6-x} x^2 y\, dy\, dx$$

$36 - 12x + x^2 - \frac{1}{4}x^2$

$$= \int_0^1 \frac{1}{2}x^2\left(25x^2 - \frac{1}{4}x^2\right) dx + \int_1^4 \frac{1}{2}x^2\left((6-x)^2 - \frac{1}{4}x^2\right) dx$$

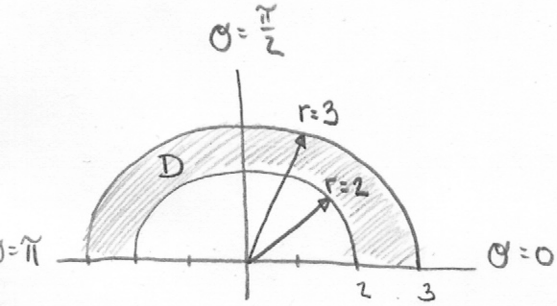
$$= \int_0^1 \frac{99}{8}x^4\, dx + \int_1^4 \left(\frac{3}{8}x^4 - 6x^3 + 18x^2\right) dx$$

$$= \frac{99}{40}\left[1^5 - 0^5\right] + \frac{3}{40}\left[4^5 - 1^5\right] - \frac{3}{2}\left[4^4 - 1^4\right] + 6\left[4^3 - 1^3\right]$$

$$= \boxed{\frac{99}{40} + \frac{3069}{40} - \frac{765}{2} + 378 = \frac{747}{10} \text{ or } 74.7}$$

10. Evaluate the following double integral over the region D.

$$\iint_D \frac{x^2 \sin(x^2+y^2)}{x^2+y^2}\, dA, \quad D = \{4 \le x^2+y^2 \le 9, y \ge 0\}$$



$D: \quad 0 \le \theta \le \pi$

$\quad\quad 2 \le r \le 3$

$$\int_0^{\pi}\int_2^3 \frac{(r\cos\theta)^2 \sin(r^2)}{r^2} \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi}\int_2^3 \cos^2\theta \sin(r^2) r\, dr\, d\theta = \int_0^{\pi} \cos^2\theta\, d\theta \int_2^3 r \sin(r^2)\, dr$$

$u = r^2$

$\frac{1}{2}du = r\,dr$

$$= \int_0^{\pi} \frac{1}{2}\left[1 + \cos(2\theta)\right] d\theta \int_4^9 \frac{1}{2}\sin(u)\, du$$

$$= \boxed{\frac{\pi}{4}\left(\cos(4) - \cos(9)\right)} \approx .202$$