Name: *ANSWER KEY* (76 POINTS) **11/1/2018**
Math 20300-ST Calculus III **Midterm Exam**

Please show all work and **box your final answers**. If you need more room, you may use the backs of the pages. Calculators are not allowed. Good luck!

1. Let P be the point $(3, -1, 2)$ And let C be the circle with equation

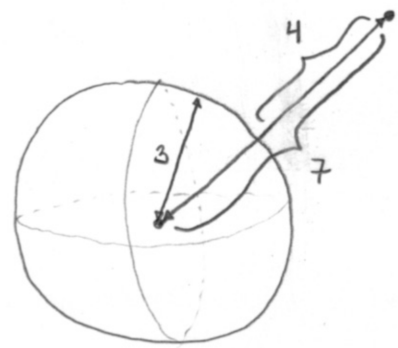
$$(x-1)^2 + (y-2)^2 + (z+4)^2 = 9. \qquad (C)$$

(a) (4 points) What is the distance between P and the center of C?

$C = (1, 2, -4)$

$d = \sqrt{(3-1)^2 + (-1-2)^2 + (2+4)^2}$

$= \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = \boxed{7}$



(b) (4 points) What is the distance between P and C?

$7 - 3 = \boxed{4}$

2. (8 points) Two intersecting lines l_1 and l_2 have the following parametric equations.

$$x = 6 - 4s, \quad y = 9 - 8s, \quad x = -4 + s \qquad (l_1)$$

$$x = 2 + 4t, \quad y = 1 + 7t, \quad x = -3 - 4t \qquad (l_2)$$

Find the angle between the lines at the point where they intersect.
Note: You do not need to find where the lines intersect and your answer may be left as a trig/inverse-trig expression.

l_1 IS PARALLEL TO $\langle -4, -8, 1 \rangle = \vec{v}_1$

l_2 IS PARALLAL TO $\langle 4, 7, -4 \rangle = \vec{v}_2$

$\therefore$ ANGLE BETWEEN l_1 & l_2 = ANGLE BETWEEN $\vec{v}_1$ & $\vec{v}_2$, θ.

$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2| \cos\theta \Rightarrow \cos\theta = \dfrac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|}$

$\cos\theta = \dfrac{\langle -4, -8, 1 \rangle \cdot \langle 4, 7, -4 \rangle}{\sqrt{4^2 + 8^2 + 1^2}\ \sqrt{4^2 + 7^2 + 4^2}} = \dfrac{-16 - 56 - 4}{9 \cdot 9} \Rightarrow \boxed{\theta = \cos^{-1}\left(\dfrac{-76}{81}\right)}$

3. (a) (8 points) Give a vector equation or parametric equations for the line through the point $(3, 1, -4)$ that is parallel to both of the following planes.

$$x - y - z = 7 \qquad (P_1)$$

$$x + y - 2z = -1 \qquad (P_2)$$

$\parallel$ to $P_1, P_2 \Rightarrow \perp$ to $\vec{n}_1, \vec{n}_2 \Rightarrow \parallel$ to $\vec{n}_1 \times \vec{n}_2$

$$\vec{n}_1 \times \vec{n}_2 = \langle 1, -1, -1 \rangle \times \langle 1, 1, -2 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ 1 & 1 & -2 \end{vmatrix} = \langle 3, 1, 2 \rangle$$

$$\therefore \boxed{\vec{\ell}(t) = \langle 3, 1, -4 \rangle + t \langle 3, 1, 2 \rangle}$$

OR

$$\boxed{\begin{aligned} x &= 3 + 3t \\ y &= 1 + t \\ z &= -4 + 2t \end{aligned}}$$

(b) (4 points) Find the point at which the line from part (a) intersects the yz-plane.

HAPPENS WHEN $x = 0 \rightarrow 3 + 3t = 0 \rightarrow t = -1$

WHEN $t = -1$, $(x, y, z) = \boxed{(0, 0, -6)}$

4. (8 points) Give a vector equation or parametric equations for the tangent line to the curve

$$\vec{r}(t) = \langle\, e^{-t} \cos t,\ e^{-t} \sin t,\ e^{-t} \,\rangle \qquad (\vec{r})$$

at the point $(1, 0, 1)$.

NOTE: $\vec{r}(0) = \langle 1, 0, 1 \rangle$.

$\therefore \vec{\ell}(t) = \vec{r}(0) + t\vec{r}\,'(0)$.

$$\vec{r}\,'(t) = \langle e^{-t}(-\cos t - \sin t), e^{-t}(-\sin t + \cos t), -e^{-t} \rangle$$

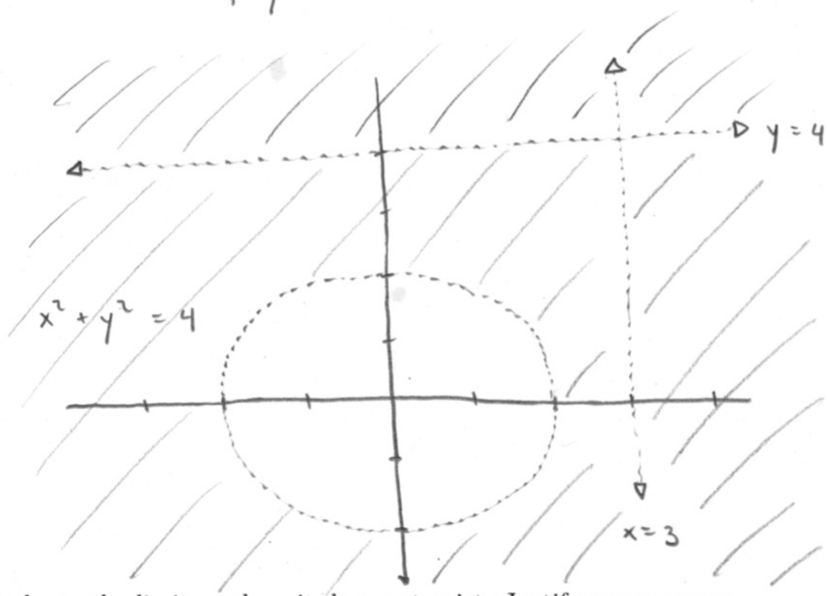
$$\vec{r}\,'(0) = \langle -1, 1, -1 \rangle$$

$$\therefore \boxed{\vec{\ell}(t) = \langle 1, 0, 1 \rangle + t \langle -1, 1, -1 \rangle} \quad \text{OR} \quad \boxed{\begin{aligned} x &= 1 - t \\ y &= t \\ z &= 1 - t \end{aligned}}$$

5. (4 points) Sketch the domain of the function $f(x, y) = \dfrac{\ln(x^2 + y^2 - 4)}{(x-3)(y-4)}$.

ARGUMENT OF $\ln$ MUST BE POS: $x^2 + y^2 - 4 > 0 \rightarrow x^2 + y^2 > 4$ (OUTSIDE THE CIRCLE)

DENOM $\neq 0 \rightarrow x \neq 3$, $y \neq 4$



6. (8 points) Evaluate the limit or show it does not exist. Justify your answer.

$$\lim_{(x,y)\to(0,0)} \frac{2x \sin y}{x^2 + y^2}$$

* ALONG $x = 0$: $\displaystyle\lim_{y \to 0} \frac{0}{y^2} = 0$ *

* ALON $y = x$: $\displaystyle\lim_{x \to 0} \frac{2x \sin x}{2x^2} = \lim_{x \to 0} \frac{\sin x}{x} = 1$ *

(YOU DO NOT NEED TO JUSTIFY THIS LIMIT, BUT IF YOU DON'T KNOW IT, APPLY L'HOSPITAL'S RULE)

SINCE THESE 2 LIMITS ARE DIFFERENT, THE LIMIT $\boxed{\text{D.N.E.}}$

7. Let $f(x,y) = \ln(2x+y)$.

(a) (4 points) Find all second partial derivatives at f.

$$f_x = \frac{2}{2x+y} \qquad f_y = \frac{1}{2x+y}$$

$$\boxed{\begin{aligned} f_{xx} &= -2(2x+y)^{-2}(2) = \frac{-4}{(2x+y)^2} \\ f_{yy} &= -(2x+y)^{-2} = \frac{-1}{(2x+y)^2} \\ f_{xy} &= f_{yx} = \frac{-2}{(2x+y)^2} \end{aligned}}$$

(b) (8 points) Find an equation for the tangent plane to the surface $z = f(x,y)$ at the point $(-1,3,0)$.

PLANE: $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$$z - 0 = \frac{2}{2(-1)+3}(x+1) + \frac{1}{2(-1)+3}(y-3)$$

$$\boxed{\begin{aligned} z &= 2(x+1) + 1(y-3) \\ \text{OR} \quad 2x + y - z &= 1 \end{aligned}}$$

8. At points (x, y, z) in a region of space for which $x^2 + y^2 \geq 1$ and $z \geq 0$ there is an electric charge

$$E(x, y, z) = z + z\ln(x^2 + y^2). \qquad (E)$$

(a) (8 points) Find the rate at which the electric charge is changing at $P(1, 0, 2)$ in the direction towards the point $Q(4, 4, 7)$.

DIRECTION : $\overrightarrow{PQ} = \langle 4-1, 4-0, 7-2 \rangle = \langle 3, 4, 5 \rangle$

$$\vec{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{1}{\sqrt{3^2+4^2+5^2}} \langle 3, 4, 5 \rangle = \frac{1}{5\sqrt{2}} \langle 3, 4, 5 \rangle$$

$$\nabla E(x,y,z) = \langle E_x, E_y, E_z \rangle = \left\langle \frac{2xz}{x^2+y^2}, \frac{2yz}{x^2+y^2}, 1 + \ln(x^2+y^2) \right\rangle$$

$$D_{\vec{u}} E(1,0,2) = \nabla E(1,0,2) \cdot \frac{1}{5\sqrt{2}} \langle 3,4,5 \rangle = \frac{\langle 4, 0, 1 \rangle \cdot \langle 3, 4, 5 \rangle}{5\sqrt{2}} = \boxed{\frac{17}{5\sqrt{2}}}$$

(b) (4 points) Find the direction of greatest increase in E at the point $P(1, 0, 2)$.

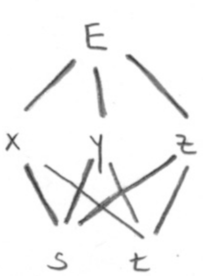
THIS IS $\nabla E(1,0,2) = \boxed{\langle 4, 0, 1 \rangle}$

OR $c\langle 4,0,1 \rangle$ WHERE $c > 0$ IS ARBITRARY

(c) (4 points) At each point (s, t) on the ground in a physics lab, the electric charge at position

$$(x, y, z) = (s + t, s - t, 2st)$$

is measured. Find the rate $\dfrac{\partial E}{\partial s}$ at which the electric charge is changing with respect to s at the point $(s, t) = (1, 1)$.



$$\frac{\partial E}{\partial s} = \frac{\partial E}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial E}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial E}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial E}{\partial s} = \left(\frac{2xz}{x^2+y^2}\right)(1) + \left(\frac{2yz}{x^2+y^2}\right)(1) + \left(1 + \ln(x^2+y^2)\right)(2t)$$

NOTE : WHEN $s = 1$, $t = 1 \Rightarrow x = 2$, $y = 0$, $z = 2$

$$\left.\frac{\partial E}{\partial s}\right|_{\substack{s=1\\ t=1\\ x=2\\ y=0\\ z=2}} = \left(\frac{8}{4}\right)(1) + (0)(1) + (1 + \ln 4)(2) = \boxed{4 + 2\ln 4}$$