MATH 212 - DRAFT FINAL EXAM B

SOLUTIONS BY JOHN ADAMSKI

1. (a) $\displaystyle\int_0^{\pi/2} t^2 \cos t \, dt$ $\qquad u = t^2 \qquad v = \sin t$

$\qquad du = 2t\,dt \qquad dv = \cos t \, dt$

$$= uv\Big|_0^{\pi/2} - \int_0^{\pi/2} v\,du = t^2 \sin t\Big|_0^{\pi/2} - \int_0^{\pi/2} 2t \sin t \, dt$$

$u = 2t \qquad v = -\cos t$

$du = 2dt \qquad dv = \sin t \, dt$

$$= t^2 \sin t\Big|_0^{\pi/2} - \Big(\underbrace{-2t\cos t\Big|_0^{\pi/2}}_{0} + \int_0^{\pi/2} 2\cos t \, dt \Big)$$

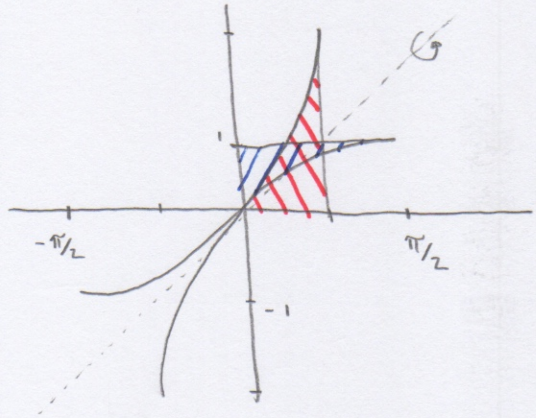
$$= \left(\frac{\pi}{2}\right)^2 - 2\sin t\Big|_0^{\pi/2} = \left(\frac{\pi}{2}\right)^2 - 2 = \boxed{\frac{\pi^2 - 8}{4}}$$

(b) $$\int \sec^3 t \tan^3 t \, dt = \int \underbrace{\sec^2 t \,(\sec^2 t - 1)}_{u^2(u^2-1)} \underbrace{\sec t \tan t \, dt}_{du}$$

$$\rightsquigarrow \int (u^4 - u^2)\, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$\rightsquigarrow \boxed{\frac{1}{5}\sec^5 t - \frac{1}{3}\sec^3 t + C}$$

(c)



$$\int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - \int_0^{\pi/2} \sin x \, dx$$

$$= \frac{\pi}{2} - 1 \quad ✓$$

$$\int_0^1 \sin^{-1} x \, dx \qquad u = \sin^{-1} x \qquad v = x$$

$$du = \frac{1}{\sqrt{1-x^2}} dx \qquad dv = dx$$

$$= \underbrace{x \sin^{-1} x \Big|_0^1}_{\frac{\pi}{2} - 0} - \underbrace{\int_0^1 \frac{x}{\sqrt{1-x^2}} dx}_{-\frac{1}{2}\int_1^0 u^{-1/2} du} \qquad u = 1 - x^2, \quad du = -2x \, dx$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_0^1 u^{-1/2} du = \frac{\pi}{2} - \left[u^{1/2} \right]_0^1 = \boxed{\frac{\pi}{2} - 1}$$

(d) $$\int \frac{t^3 - 1}{t^3 + 1} dt = \int \frac{(t^3+1) - 2}{t^3 + 1} dt$$

$$= \int 1 - \frac{2}{t^3 + 1} dt = \int 1 + \frac{-2}{(t+1)(t^2-t+1)} dt$$

$$\uparrow$$

$$t^3 + 1 = (t+1)(t^2 - t + 1)$$

$$= \int 1 + \frac{A}{t+1} + \frac{Bt + C}{t^2 - t + 1}\, dt$$

WHERE $-2 = A(t^2 - t + 1) + (Bt + C)(t+1)$

$$= At^2 - At + A + Bt^2 + Bt + Ct + C$$

$$-2 = (A+B)t^2 + (-A+B+C)t + (A+C)$$

$$\Rightarrow \quad A + B = 0 \quad (1)$$

$$-A + B + C = 0 \quad (2)$$

$$A + C = -2 \quad (3)$$

$(2) - (1) - (3)$: $-3A = 2 \Rightarrow A = -2/3$

$\Rightarrow B = 2/3$

$\Rightarrow C = -4/3$

$$= \int 1\, dt - \frac{2}{3}\int \frac{1}{t+1}\, dt + \int \frac{\frac{2}{3}t - \frac{4}{3}}{t^2 - t + 1}\, dt$$

$$= t - \frac{2}{3}\ln|t+1| + \frac{2}{3}\int \frac{t - 2}{\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}}\, dt \qquad u = t - \frac{1}{2}, \quad du = dt$$

$$= t - \frac{2}{3}\ln|t+1| + \frac{2}{3}\int \frac{u + 3/2}{u^2 + 3/4}\, du$$

$$= t - \frac{2}{3}\ln|t+1| + \frac{2}{3}\left[\int \frac{u}{u^2 + 3/4}\, du + \frac{3}{2}\int \frac{1}{u^2 + 3/4}\, du\right]$$

$$= t - \frac{2}{3}\ln|t+1| + \frac{1}{3}\ln\left|u^2 + \frac{3}{4}\right| + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2u}{\sqrt{3}} + C$$

$$= \boxed{t - \frac{2}{3}\ln(t+1) + \frac{1}{3}\ln\left|t^2 - t + 1\right| + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\left(t-\frac{1}{2}\right)}{\sqrt{3}} + C}$$

2. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{3n^2+n+1}$ THE SERIES IS ALTERNATING & THE MAGNITUDE OF THE TERMS GOES TO ZERO. $\therefore$ THE SERIES CONVERGES.

DOES IT CONVERGE ABSOLUTELY?

$\displaystyle\sum_{n=1}^{\infty} \frac{n}{3n^2+n+1}$ LIMIT COMPARISON TEST:

$$\lim_{n\to\infty} \frac{\frac{n}{3n^2+n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{3n^2+n+1} = \frac{1}{3}$$

SINCE THE LIMIT IS FINITE & $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES,

THE LIMIT COMPARISON THM IMPLIES THAT THE SERIES ALSO DIVERGES.

$\therefore$ THE SERIES $\boxed{\text{CONVERGES CONDITIONALLY.}}$

(b) $$\sum_{n=2}^{\infty} \frac{2^n+5}{3^n} = \sum_{n=2}^{\infty}\left(\frac{2}{3}\right)^n + 5\sum_{n=2}^{\infty}\left(\frac{1}{3}\right)^n$$

THESE ARE BOTH GEOMETRIC SERIES $\left(\displaystyle\sum_{n=1}^{\infty} ar^{n-1}\right)$

WITH $|r| < 1$.

$\Rightarrow$ THEY BOTH CONVERGE.

$\Rightarrow$ $\boxed{\text{ORIGINAL SERIES CONVERGES (ABSOLUTELY SINCE ALL TERMS ARE POSITIVE).}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n\sqrt{n}}$ ← NOTE THAT THE TERMS ARE SOMETIMES POSITIVE & SOMETIMES NEGATIVE, BUT NOT ALTERNATING.

THEREFORE WE CHECK FOR ABSOLUTE CONVERGENCE.

$$0 \le \left| \frac{\cos(n^2)}{n\sqrt{n}} \right| \le \frac{1}{n^{3/2}}$$

↑

SINCE $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ CONVERGES BY p-TEST

(WITH $p = \frac{3}{2} > 1$)

THE DIRECT COMPARISON TEST IMPLIES THE ORIGINAL SERIES

$\boxed{\text{CONVERGES ABSOLUTELY.}}$

3. (a) ALONG $y = 0$: $\displaystyle\lim_{x \to 0} \frac{\cancel{x^2}\cos(x)}{\cancel{x^2}} = 1$

ALONG $x = 0$: $\displaystyle\lim_{y \to 0} \frac{0}{y^2} = 0$

SINCE THERE ARE DIFFERENT LIMITS ALONG DIFFERENT PATHS TO $(0,0)$, THE LIMIT DOES NOT EXIST.