$$\int \underbrace{(\text{ALL } \sec x)}_{f(u)} \underbrace{\sec x \tan x \, dx}_{du} \quad \star$$

ex. $\int \sec^6 3x \tan 3x \, dx$

$$\int \underbrace{(\text{ALL } \tan)}_{f(u)} \underbrace{\sec^2 x \, dx}_{du} \quad \star\star$$

$\star$

$$\int \sec^5 3x \sec 3x \tan 3x \, dx$$

Let $u = \sec 3x$

$$\frac{du}{3} = \frac{\cancel{3} \sec 3x \tan 3x \, dx}{\cancel{3}}$$

$$\frac{1}{3} \int u^5 \, du = \frac{1}{18} u^6 + C$$

$$\frac{1}{18} \sec^6 (3x) + C$$

$\star\star$

$$\int \sec^4 3x \tan 3x \sec^2 3x \, dx$$

$$\sec^4 3x = (\sec^2 3x)^2 = (\tan^2 3x + 1)^2$$

$$\int \underbrace{(\tan^2 3x + 1)^2 \tan 3x}_{u = \tan 3x} \underbrace{\sec^2 3x \, dx}_{\frac{1}{3} du}$$

$$\frac{du}{3} = \frac{\cancel{3} \sec^2 3x \, dx}{\cancel{3}}$$

$$\frac{1}{3}\int (u^2+1)^2 u \, du = \frac{1}{3}\int u^5 + 2u^3 + u \, du$$

$$(u^4+2u^2+1)u$$

$$= \frac{1}{3}\left[\frac{1}{6}u^6 + \frac{1}{2}u^4 + \frac{1}{2}u^2\right] + C$$

$\sim \; = \frac{1}{\;} \ldots$

## §8.5 INTEGRATION OF RATIONAL FUNCTIONS WITH PARTIAL FRACTIONS

$$\int \frac{P(x)}{Q(x)}\,dx \quad , \quad P, Q \text{ POLYNOMIALS.}$$

$\boxed{1.}$ IF $\overbrace{\text{DEG}(P) \geq \text{DEG}(Q)}^{\text{IMPROPER}}$ USE POLYNOMIAL LONG DIVISION

TO REWRITE AS $\displaystyle\int f(x) + \frac{R(x)}{Q(x)}\,dx$

POLYNOMIAL

PROPER RATIONAL FUNCTION.

$\boxed{2.}$ FACTOR DENOMINATOR $Q(x)$ COMPLETELY.

e.g. $Q(x) = (x-r_1)(x-r_2)(x-r_3)^2$

(DEGREE 4 POLYNOMIAL)

$\boxed{3.}$ WRITE DOWN THE FORM OF THE PARTIAL FRACTION DECOMPOSITION FOR $\dfrac{R(x)}{Q(x)}$.

FACT: (FUNDAMENTAL THEOREM OF ALGEBRA)

EVERY POLYNOMIAL CAN BE FACTORE INTO

LINEAR FACTORS & IRREDUCIBLE QUADRATIC FACTORS

$(ax+b)$

$(ax^2+bx+c)$

DISCRIMINANT: $b^2-4ac < 0$

$$x = \frac{-b \pm \sqrt{\boxed{b^2-4ac}}}{2a}$$

e.g. $Q(x) = \underset{①}{(ax+b)^n}\ \underset{②}{(ax^2+bx+c)^m} \ldots$ (MORE POSSIBLY)

I. FOR <u>EVERY</u> LINEAR FACTOR OF $Q(x)$, WITH EXPONENT $n$,

THE PARTIAL FRACTION DECOMPOSITION (PFD) OF $\frac{R(x)}{Q(x)}$

CONTAINS $n$ TERMS:

$(x-a)^n$ IS A FACTOR OF $Q(x)$

$$\Rightarrow \quad \frac{R(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_n}{(x-a)^n}$$

WHERE $A_1, \ldots, A_n$ ARE YET TO BE DETERMINED CONSTANTS.

ex. $\dfrac{7x+1}{x^2+6x+9} = \dfrac{7x+1}{(x+3)^2}$ ← FACTOR $Q(x)$

LINEAR FACTOR REPEATED TWICE.

PFD

$$\frac{7x+1}{(x+3)^2} = \frac{A_1}{x+3} + \frac{A_2}{(x+3)^2}$$

CONSTANTS

$$\int \frac{7x+1}{(x+3)^2}\,dx = \int \frac{A_1}{x+3} + \frac{A_2}{(x+3)^2}\,dx$$ FORM

$$= \boxed{A_1 \ln|x+3| - A_2(x+3)^{-1} + C}$$

5 CONSTANTS

ex. PFD

$$\frac{5x^3-2}{\underbrace{(x-1)^2}\,\underbrace{(x-3)^3}} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1}{x-3} + \frac{B_2}{(x-3)^2} + \frac{B_3}{(x-3)^3}$$

$(x^2+bx+c)$, $(b^2-4ac<0)$

## II. FOR EACH IRREDUCIBLE QUADRATIC FACTOR OF $Q(x)$, REPEATED $n$ TIMES, THE PFD OF $\frac{R(x)}{Q(x)}$ HAS $n$ TERMS:

$$Q(x) = (x^2+bx+c)^n$$

$$\frac{R(x)}{Q(x)} = \frac{A_1x+B_1}{(x^2+bx+c)} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \dots + \frac{A_nx+B_n}{(x^2+bx+c)^n}$$

2 $\qquad$ n

**e.x.**

$$\frac{2x+4}{(x-3)^2(x^2+x+10)^2} = \frac{A_1}{x-3} + \frac{A_2}{(x-3)^2} + \frac{B_1x+C_1}{x^2+x+10} + \frac{B_2x+C_2}{(x^2+x+10)^2}$$

$$b^2-4ac = 1-4(+10) < 0$$

1 TERM FOR EVERY FACTOR

**ex.**

$$\frac{1}{(x-2)^2(x^2+2x+20)} \overset{PFD}{=} \underbrace{\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+2x+20}}$$

**ex.** GIVE THE FORM OF THE PFD FOR $\frac{3x^2-1}{(x-1)(x-2)^2(x^2+3x+15)}$

$$= \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+3x+15}$$

ex.

$$\frac{1}{(x-1)^{2}(x^2+x+8)^3} \overset{\text{PFD}}{=} \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+8} + \frac{Ex+F}{(x^2+x+8)^2} + \frac{Gx+H}{(x^2+x+8)^3}$$

SOLVING FOR THE UNKNOWN CONSTANTS A, B, C, ETC.

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$$\int \frac{x+3}{2x^3-8x}\,dx = \int \frac{x+3}{2x(x+2)(x-2)}\,dx$$

$2x(x^2-4)$ $a=1$, $b=0$, $c=-4$ $\quad b^2-4ac = -4(1)(-4) = 16 > 0$ REDUCIBLE!

$$= \int \frac{A}{2x} + \frac{B}{x+2} + \frac{C}{x-2}\,dx$$

TRUE FOR ALL X! (IDENTITY)

$$\frac{x+3}{2x(x+2)(x-2)} = \frac{A}{2x} + \frac{B}{x+2} + \frac{C}{x-2}$$

ALGEBRA.

MULTIPLY EVERY TERM BY $\underbrace{2x(x+2)(x-2)}_{\text{LCD}}$ : CANCELS WITH EVERY SINGLE DENOM.

~~$x+3 = A(x+2)(x-2) + B(2x)(x-2) + C(2x)(x+2)$~~

$$\frac{x+3}{\cancel{2x}(x+\cancel{2})(x-\cancel{2})} \cdot \cancel{2x}(x+\cancel{2})(x-\cancel{2}) = \frac{A}{\cancel{2x}} \cdot \cancel{2x}(x+2)(x-2) + \frac{B}{\cancel{x+2}} \cdot 2x(\cancel{x+2})(x-2)$$

$$+ \frac{C}{\cancel{x-2}} \cdot 2x(x+2)(\cancel{x-2})$$

$$x+3 = A(x+2)(x-2) + B(2x)(x-2) + C(2x)(x+2)$$

SOLVE FOR A, B, C.

TRUE FOR ALL $x$! SO PLUG IN $\underbrace{\text{CONVENIENT VALUES}}$

VALUES THAT MAKE SOME FACTORS EQUAL 0.

$x=0$: $\quad 0+3 = A(0+2)(0-2) + \underbrace{B(\overset{0}{2\cdot 0})(0-2)}_{0} + \underbrace{C(\overset{0}{2\cdot 0})(0+2)}_{0}$

$$3 = -4A \Rightarrow \boxed{A = -\frac{3}{4}}$$

$x=2$: $\quad 5 = 16C \Rightarrow \boxed{C = \frac{5}{16}}$

$x=-2$: $\quad 1 = 16B \Rightarrow \boxed{B = \frac{1}{16}}$

$$\int \frac{x+3}{2x(x+2)(x-2)}\,dx = \int \frac{-3/4}{2x} + \frac{1/16}{x+2} + \frac{5/16}{x-2}\,dx$$

$$= \boxed{-\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C}$$

ex. $\int \frac{x^2+2x+1}{(x^2+1)^2}\,dx$

$x^2+1$ IS IRREDUCIBLE QUADRATIC

$a=1,\ b=0,\ c=1 \qquad b^2-4ac<0$

$$\left[\frac{x^2+2x+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}\right](x^2+1)^2$$

NEVER 0

$$x^2+2x+1 = (Ax+B)(x^2+1) + Cx + D$$

1ST METHOD WILL NOT WORK → NO CHOICE OF VALUE FOR $x$ WILL MAKE FACTOR $x^2+1=0$.

2nd METHOD: EXPAND & REGROUP RIGHT SIDE OF EQ & SET POLYNOMIAL COEFFICIENTS EQUAL TO EACH OTHER.

$$x^2+2x+1 = (Ax+B)(x^2+1) + Cx + D$$

$$= Ax^3 + Ax + Bx^2 + B + Cx + D$$

$A=0$

$$0x^3 + x^2 + 2x + 1 = Ax^3 + Bx^2 + (A+C)x + (B+D)$$

$B=1$

TRUE FOR ALL VALUES OF $x$

IF & ONLY IF COEFF. ARE EQUAL.

$$A + C = 2 \xrightarrow{A=0} C = 2$$

$$B + D = 1 \xrightarrow{B=1} D = 0$$

$A = 0$
$B = 1$
$C = 2$
$D = 0$

$$\int \frac{x^2+2x+1}{(x^2+1)^2}\,dx = \int \frac{\overset{0x+1}{Ax+B}}{x^2+1} + \frac{\overset{2x+0}{Cx+D}}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x^2+1} + \frac{2x}{(x^2+1)^2}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$$= \boxed{\tan^{-1} x - \frac{1}{x^2+1} + C}$$

ex. $$\int \frac{1}{\underline{\underline{(x+5)^2}}\,\underline{\underline{(x-1)}}}\,dx = \int \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{x-1}\,dx$$

$$\frac{1}{\cancel{(x+5)^2}\cancel{(x-1)}} \cdot \cancel{(x+5)^2}\cancel{(x-1)} = \frac{A}{\cancel{x+5}}(x+5)^{\cancel{2}}(x-1) + \frac{B}{\cancel{(x+5)^2}}\cancel{(x+5)^2}(x-1)$$

$$+ \frac{C}{\cancel{x-1}} \cdot (x+5)^2\cancel{(x-1)}$$

$$1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$$

$x = 1$ : $\quad 1 = 36C \longrightarrow C = \frac{1}{36}$

$x = -5$ : $\quad 1 = -6B \longrightarrow B = -\frac{1}{6}$

$x = 0$ : $\quad 1 = -5A + \left(-\frac{1}{6}\right)(-1) + \left(\frac{1}{36}\right)(25)$

ARBITRARY

$$1 = -5A + \frac{1}{6} + \frac{25}{36} = -5A + \frac{31}{36}$$

$$\frac{5}{36} = -5A \longrightarrow A = -\frac{1}{36}$$

$-\frac{1}{6}\int \frac{1}{u^2}\,du$

$u = x+5$

$$\int \frac{1}{(x+5)^2(x-1)}\,dx = \int \frac{-1/36}{x+5} + \frac{-1/6}{(x+5)^2} + \frac{1/36}{x-1}\,dX$$

$$= \boxed{-\frac{1}{36}\ln|x+5| + \frac{1}{6}(x+5)^{-1} + \frac{1}{36}\ln|x-1| + C}$$