§8.7 NUMERICAL INTEGRATION

TILL NOW: INTEGRATION TECHNIQUES

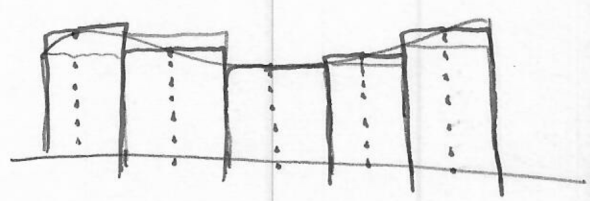
REALITY: MOST INTEGRALS DO NOT HAVE ELEMENTARY SOLUTIONS.

ex. $\displaystyle\int_0^3 e^{x^2}\,dx = \overset{?}{} \; F(x)\Big|_0^3$

IF WE CAN'T EVALUATE EXACTLY, WE APPROXIMATE IT

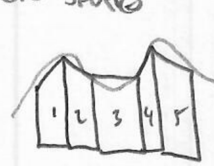
- RIGHT RIEMANN SUMS
- LEFT RIEMANN SUMS
- MIDPOINT RIEMAN SUMS

$\Big\}$ RECTANGLES

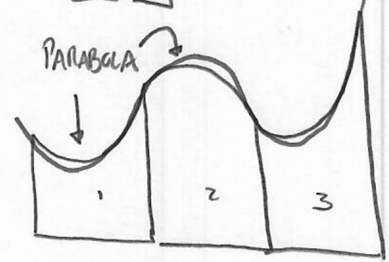


NEW: APPROXIMATE WITH OTHER SHAPES

1. TRAPEZOIDS



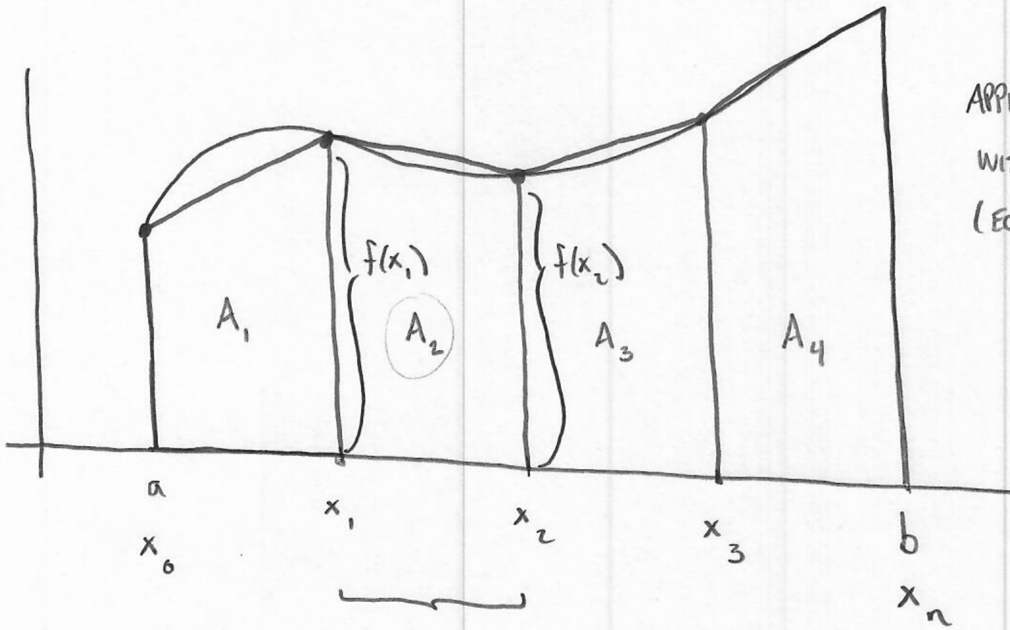
2. PARABOLIC ARCS



DEFINITE INTEGRAL

1. TRAPEZOID RULE

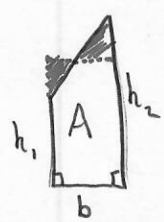
APPROXIMATE $\displaystyle\int_a^b f(x)\,dx$



APPROXIMATE AREA WITH n TRAPEZOIDS (EQUAL WIDTH Δx)

LENGTH OF EACH SUBINTERVAL $\quad \Delta x = \dfrac{b-a}{n}$ ← # SUBINTERVALS (# TRAPEZOIDS)

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} A_i$$



$$A = \frac{1}{2}(h_1 + h_2)\,b$$

$$= \boxed{\sum_{i=1}^{n} \frac{1}{2}\Big(f(x_{i-1}) + f(x_i)\Big)\Delta x}$$ ← TRAPEZOID RULE

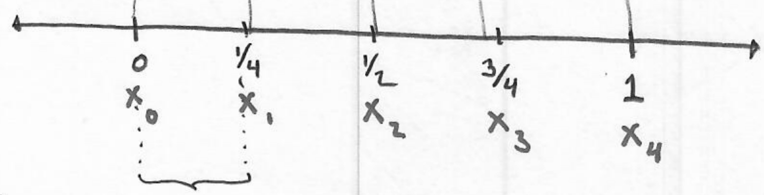
$$= \frac{\Delta x}{2}\Big(\overbrace{f(x_0) + f(x_1)}^{i=1} + \overbrace{f(x_1) + f(x_2)}^{i=2} + \ldots + \overbrace{f(x_{n-1}) + f(x_n)}^{i=n}\Big)$$

* PLUG IN #'s TO f (INTEGRAND)

$$= \boxed{\frac{\Delta x}{2}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\Big)}$$

ex. $\int_0^1 e^{x^2}\,dx$ USE TRAP. RULE WITH $n=4$ TO APPROXIMATE.

$n=4$ SUBINTERVALS

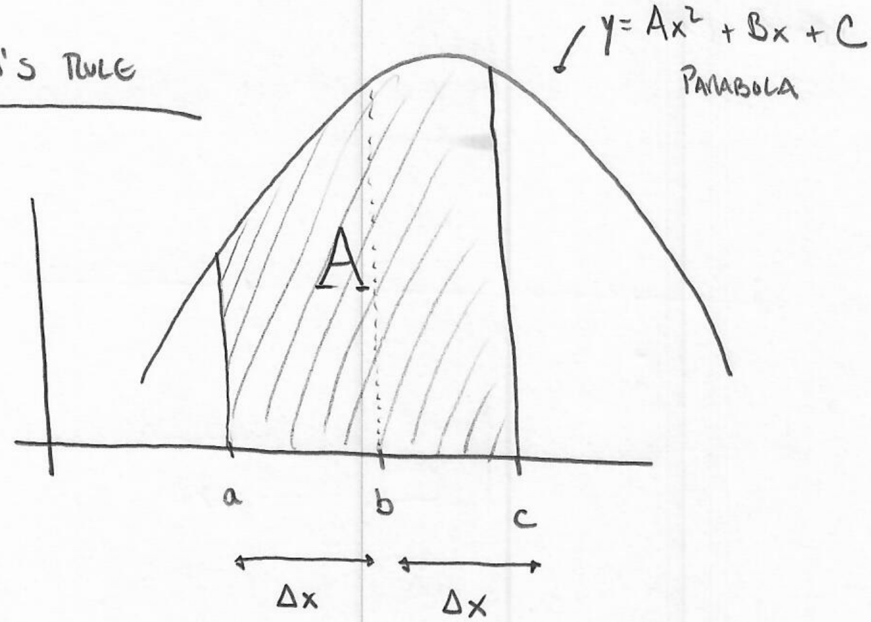


$f(x)=e^{x^2}$ $\qquad \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

$$\int_0^1 e^{x^2}\,dx \approx \frac{\Delta x}{2}\left(f(x_0)+\underline{2f(x_1)+2f(x_2)+2f(x_3)}+f(x_4)\right)$$

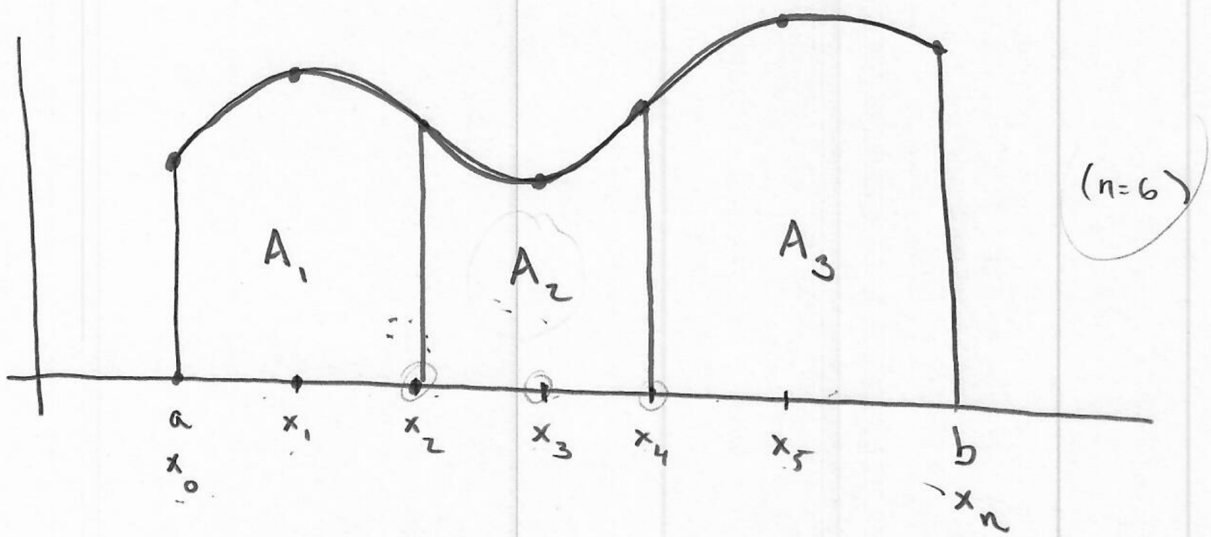
$$= \boxed{\frac{1}{8}\left(1+2e^{1/16}+2e^{1/4}+2e^{9/16}+e\right)}$$

2. SIMPSON'S RULE



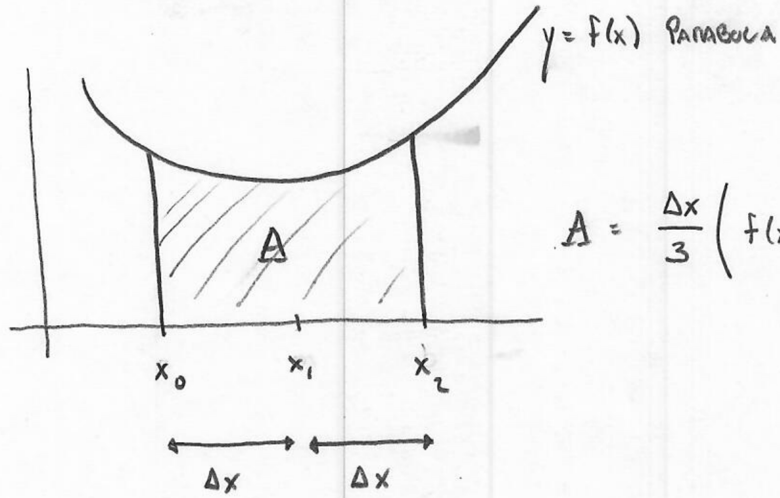
FACT: $A = \frac{\Delta x}{3}\left(f(a) + 4f(b) + f(c) \right)$ (PROOF IN TXT)

$\int_a^b f(x)\,dx$ SUBDIVIDE $[a, b]$ INTO n SUBINTERVALS WITH n EVEN



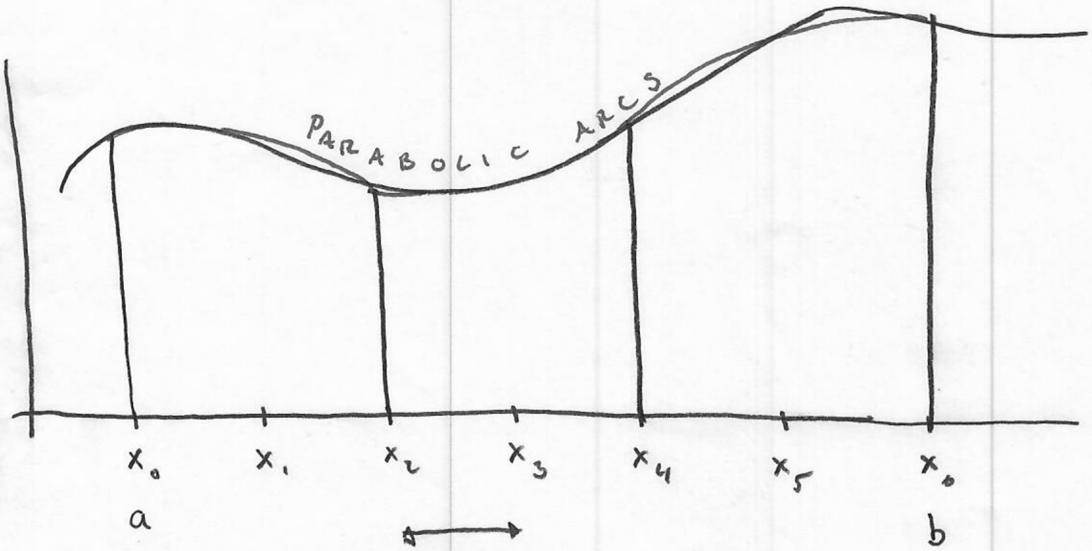
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n/2} A_i$$

SIMPSON'S RULE:



$$A = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + f(x_2) \right)$$

(n EVEN)



$$\Delta x = \frac{b-a}{n}$$

$$S = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + f(x_2) \right) + \frac{\Delta x}{3}\left(f(x_2) + 4f(x_3) + f(x_4) \right) + \ldots$$

$$+ \frac{\Delta x}{3}\left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right)$$

$$S = \frac{\Delta x}{3}\left(f(x_0) + \underbrace{4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 4f(x_{n-1})}_{} + f(x_n) \right)$$

1   4, 2, 4, 2, ..., 2 4   1

$$= \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

$$+ \frac{\Delta x}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right) + \dots$$

$$+ \frac{\Delta x}{3}\left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

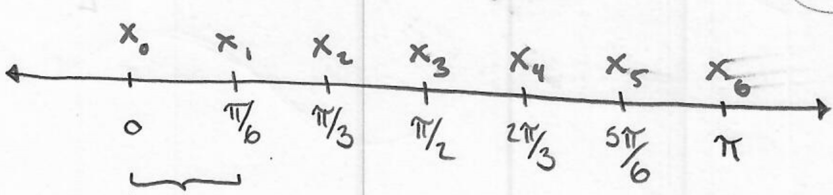
ALTERNATING 4, 2, 4, 2, ..., 2, 4

$$= \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \dots$$

$$+ 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \Big)$$

$f(x) = \sin(x^2)$

ex. $\displaystyle\int_0^{\pi} \sin(x^2)\,dx$ APPROXIMATE WITH SIMPSON'S RULE, $n = 6$ ← EVEN



$$\Delta x = \frac{b-a}{n} = \frac{\pi}{6}$$

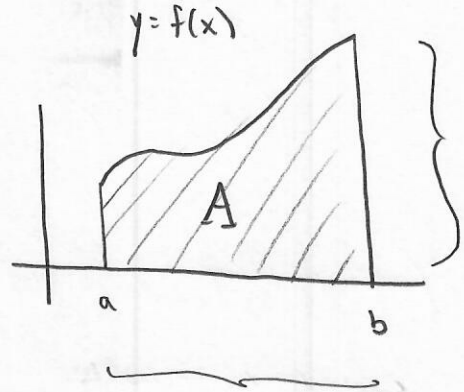
$$\approx \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right)$$

$$= \frac{\pi}{18}\left(0 + 4\sin\left(\frac{\pi^2}{36}\right) + 2\sin\left(\frac{\pi^2}{9}\right) + 4\sin\left(\frac{\pi^2}{4}\right) + 2\sin\left(\frac{4\pi^2}{9}\right)\right.$$

$$\left. + 4\sin\left(\frac{25\pi^2}{36}\right) + \sin\left(\pi^2\right)\right)$$

§8.8 IMPROPER INTEGRALS

$$\int_a^b f(x)\,dx \;=\; \text{AREA OF REGION}$$



II. FINITE HEIGHT

f IS BOUNDED ON (a, b)

I. FINITE WIDTH

a, b BOTH FINITE

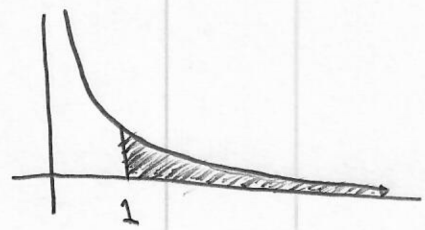
I. + II. $\Rightarrow$ PROPER INTEGRALS.

WHAT ABOUT ...

I. INFINITE WIDTH

(IMPROPER OF TYPE I.)

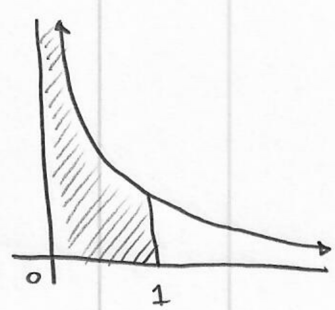
e.g. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$



II. INFINITE HEIGHT

(IMPROPER OF TYPE II.)

e.g. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$



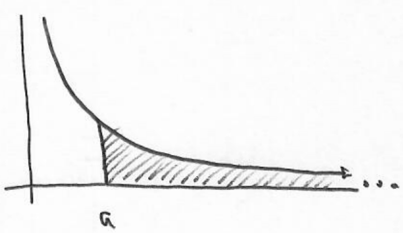
I. IMPROPER INTEGRALS OF TYPE I.

IF $f(x)$ IS CONTINUOUS ON $[a, \infty)$ THEN

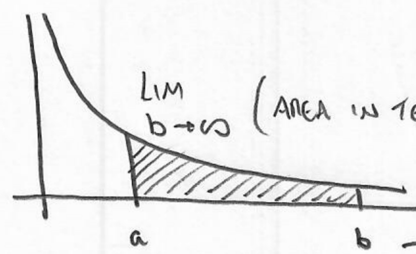
$\int_a^\infty f(x)\,dx$ IS DEFINED TO BE

$$\lim_{b \to \infty} \int_a^b f(x)\,dx$$

- IF THIS LIMIT EXIST WE SAY THE IMPROPER INTEGRAL <u>CONVERGES</u>.
- OTHERWISE, WE SAY IT <u>DIVERGES</u>.



$=$



$\lim_{b \to \infty}$ (AREA IN TERMS OF b)

b ⟶ SLIDE TO ∞.

VARIABLE

IN SUMMARY,

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

SIMILARLY,

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx \quad , \text{ AND}$$

$$\int_{-\infty}^\infty f(x)\,dx = \underbrace{\int_{-\infty}^t f(x)\,dx + \int_t^\infty f(x)\,dx}$$

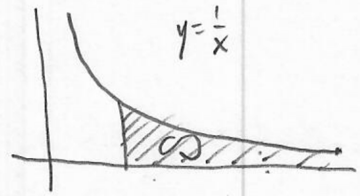
WHERE t IS ANY FINITE NUMBER.

* IF EITHER OF THESE LIMITS DIVERGES, THEN THE WHOLE THING DIVERGES.

ex.

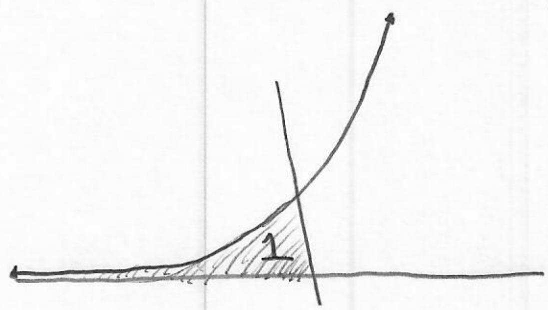
$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln b - \overset{0}{\ln 1}\right]$$

$$= \lim_{b\to\infty} \ln b = \infty$$



ex.

$$\int_{-\infty}^{0} e^x\,dx$$

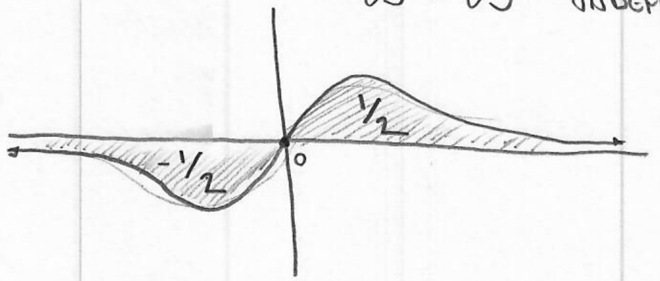


$$= \lim_{a\to-\infty}\int_a^0 e^x\,dx = \lim_{a\to-\infty}\left(e^0 - e^a\right)$$

$$= 1 - \left(\underbrace{\lim_{a\to-\infty} e^a}_{0}\right) = \boxed{1}$$

ex. $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2}\,dx$

$\infty - \infty$ UNDEFINED.



$$= \int_{-\infty}^{0} x e^{-x^2}\,dx + \int_{0}^{\infty} x e^{-x^2}\,dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} x e^{-x^2}\,dx + \lim_{b \to \infty} \int_{0}^{b} x e^{-x^2}\,dx$$

$u = -x^2$

$du = -2x\,dx \longrightarrow -\frac{1}{2}\,du = x\,dx$

$-\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u \rightsquigarrow -\frac{1}{2}e^{-x^2}$

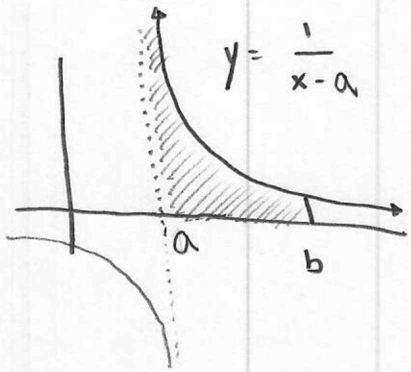
$$= \lim_{a \to -\infty}\left(-\frac{1}{2}e^0 + \frac{1}{2}e^{-a^2}\right) + \lim_{b \to \infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}e^0\right)$$

$$= \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = \boxed{0}$$

II. IMPROPER INTEGRALS OF TYPE II.

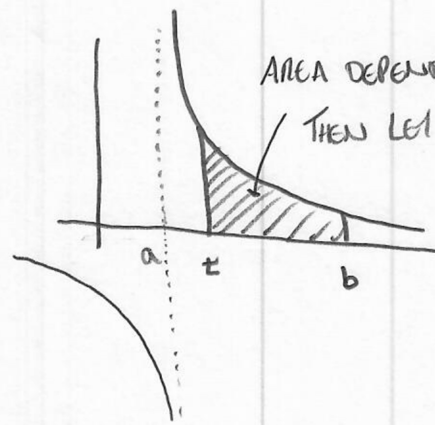
• IF f IS CONTINUOUS ON $(a, b]$

AND DISCONTINUOUS AT a.



THEN $\int_a^b f(x)\,dx$ IS DEFINED TO BE

$$= \lim_{t \to a^+} \int_t^b f(x)\,dx$$



SUMMARY, IF f HAS DISCONTINUITY AT a,

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

, SIMILARLY,

IF f HAS DISCONTINUITY AT b,

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

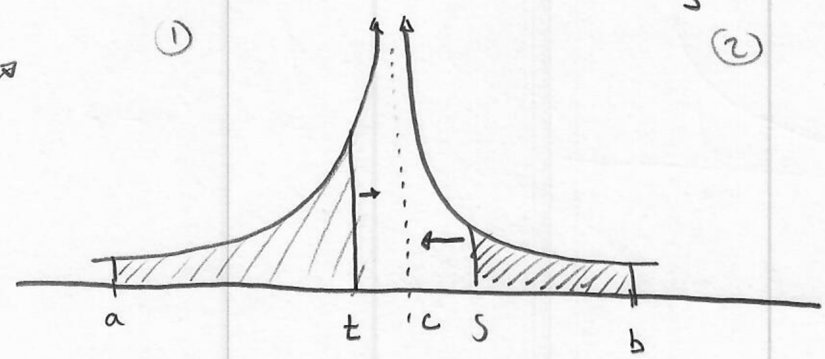
, AND

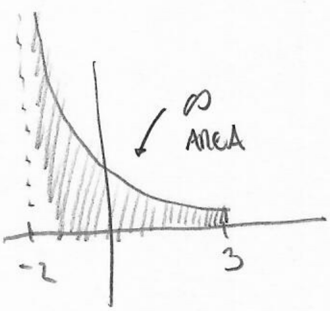
IF f IS CONTINUOUS ON $[a, c) \cup (c, b]$

WITH A DISCONTINUITY AT c, THEN

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \underbrace{\lim_{t \to c^-} \int_a^t f(x)\,dx}_{①} + \underbrace{\lim_{s \to c^+} \int_s^b f(x)\,dx}_{②}$$





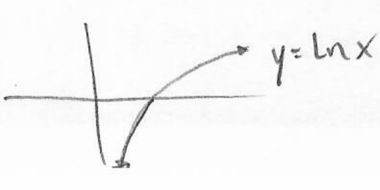
IF EITHER OF THESE LIMITS DOES NOT EXIST

THEN THE IMPROPER INTEGRAL DIVERGES.

ex. $\displaystyle\int_{-2}^{3} \frac{1}{x+2}\,dx$ IMPROPER BECAUSE $f(x) = \dfrac{1}{x+2}$

IS DISCONTINUOUS AT LEFT-END POINT $x = -2$

$$= \lim_{t \to -2^+} \int_t^3 \frac{1}{x+2}\,dx = \lim_{t \to -2^+} \Big(\ln|3+2| - \ln|t+2| \Big)$$



$$= \ln 5 - \lim_{t \to -2^+} \ln|\underbrace{t+2}_{\downarrow\, 0^+}| = \ln 5 - (-\infty)$$

$$= \infty$$

ex. $$\int_{-5}^{3} \frac{1}{x+2}\,dx$$ IMPROPER : DISCONTINUOUS AT $-2 \in [-5, 3]$

$$= \underbrace{\int_{-5}^{-2} \frac{1}{x+2}\,dx + \underbrace{\int_{-2}^{3} \frac{1}{x+2}\,dx}_{\text{DIVERGES}}}$$

THEREFORE, THE WHOLE IMPROPER INTEGRAL <u>DIVERGES</u>.

HW. UP TO 8.7 (INCLUDING 8.7)

BEGIN 8.8 (BUT WE WILL CONTINUE 8.8 ON MONDAY, - p-TEST
- COMPARISON TEST)

Quiz #2 $\boxed{8.3-5\ ,\ 8.7}$ TUE. 6/16

8.3 8.4 8.5 8.7

IMPROPER INTEGRALS ARE LIMITS

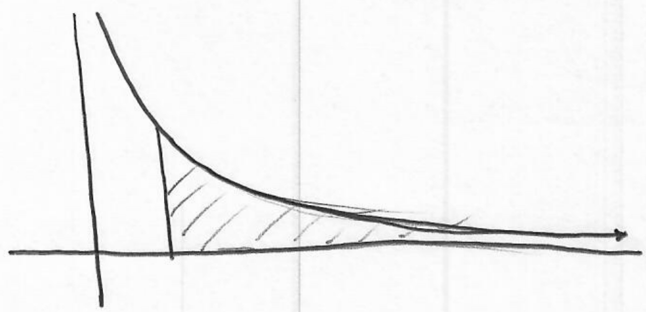
e.g $$\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\left(\int_1^t f(x)\,dx\right)$$

LIMIT EXISTS

IMP. INT. <u>CONVERGES</u>

LIMIT D.N.E.

IMP INT <u>DIVERGES</u>



MW 1:30 - 3:30 PM OFFICE HOURS