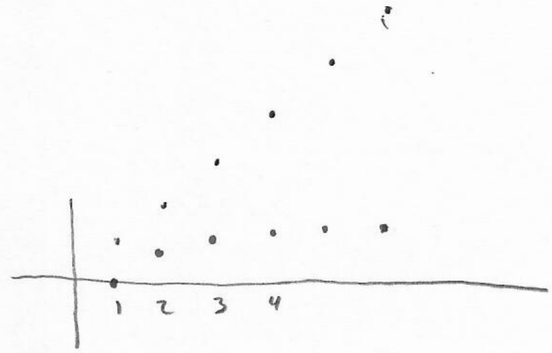
§10.1 SEQUENCES

COMMON LIMITS OF SEQUENCES:

1. $\boxed{\lim_{n\to\infty} \frac{\ln n}{n} = 0}$ ← a_n



Let $f(x) = \frac{\ln x}{x}$, Dom $(f) = (0, \infty)$

$f(n) = a_n$ n^{TH} TERM OF SEQ.

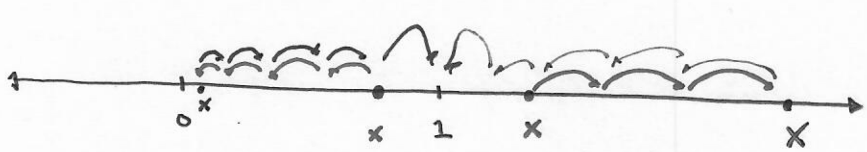
$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) \overset{L'Ho}{=} \ldots = 0$$

2. $\lim_{n\to\infty} x^{1/n} = 1$, $x > 0$ FIXED.

e.g. $x = 2$: $2, 2^{1/2}, 2^{1/3}, 2^{1/4}, \ldots \sqrt[n]{2}, \ldots$

$\to 1$

$x = \frac{1}{2}$: $\frac{1}{2}, \sqrt{\frac{1}{2}}, \sqrt[3]{\frac{1}{2}}, \ldots \to 1$



$x, x^2, x^3, \ldots$

$x, x^{1/2}, x^{1/3}, x^{1/4}$

3. $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ (SAME TRICK AS FOR 1 (L'HÔ))

4. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

e.g. $1,\ 2^{1/2},\ 3^{1/3},\ 4^{1/4},\ \dots \to 1$

$f(x) = x^{1/x}$, $a_n = f(n)$.

$\lim_{x\to\infty} x^{1/x}$: ∞^0 IND FORM

$$= \lim_{x\to\infty} e^{\ln\left(x^{1/x}\right)} \overset{\text{CONT.}}{=} \lim_{x\to\infty} e^{\frac{\ln x}{x}}$$

$$= e^{\lim_{x\to\infty}\frac{\ln x}{x}} : \frac{\infty}{\infty} \overset{\text{L'HÔ}}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{x}\to 0}{1\to 1}} = e^0 = 1 \checkmark$$

PROOF OF #3: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ (FOR ANY x) $= e^x$

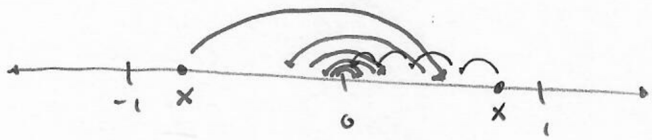
1^∞ IND. FORM

$$= \lim_{y\to\infty} e^{\ln\left(\left(1+\frac{x}{y}\right)^y\right)}$$

$$= \lim_{y\to\infty} e^{\boxed{y\ln\left(1+\frac{x}{y}\right)}} : \infty\cdot 0 = e^{\lim_{y\to\infty}\boxed{\frac{\ln\left(1+\frac{x}{y}\right)}{y^{-1}}}} : \frac{0}{0}$$

$$= e^{\lim_{y\to\infty}\frac{\frac{1}{1+\frac{x}{y}}\cdot\left(\cancel{-}x\cancel{y^{-2}}\right)}{\cancel{-y^{-2}}}} = e^{\lim_{y\to\infty}\frac{x}{1+\frac{x}{y}\to 0}} = e^x \checkmark$$

5. $\lim_{n \to \infty} x^n = 0, \quad |x| < 1 \quad (-1 < x < 1)$



6. $\lim_{n \to \infty} \frac{x^n}{n!} = 0$ (FOR ANY x).

" $n!$ GROWS FASTER THAN ANY POLYNOMIAL GROWTH "

$a_1 = \frac{x}{1} \qquad \boxed{a_2 = \frac{x^2}{2 \cdot 1} = \frac{x}{2} \cdot \frac{x}{1} = \frac{x}{2} \cdot a_1}$

$a_3 = \frac{x^3}{3!} = \frac{x \cdot x^2}{3 \cdot 2!} = \frac{x}{3} \cdot a_2$

$a_4 = \frac{x^4}{4!} = \frac{x \cdot x^3}{4 \cdot 3!} = \frac{x}{4} a_3$

$4! = 4(3 \cdot 2 \cdot 1) = 4 \cdot 3!$

$\vdots$

$a_n = \frac{x}{n} \cdot a_{n-1}$ $\quad x$ IS FIXED. n IS INCREASING.

EVENTUALLY, $n > x$, AND $\frac{x}{n} < 1$.

THAT MEANS $a_n < a_{n-1}$ (SMALLER THAN PREVIOUS TERM!)

IN PARTICULAR, WHEN $n > 2x$, WE HAVE $\left(\frac{x}{n} < \frac{1}{2} \right)$

$a_n = \frac{x}{n} a_{n-1} < \frac{1}{2} a_{n-1}$

$$\Rightarrow a_{n+1} < \left(\frac{1}{2}\right)^2 a_{n-1}$$

$$a_{n+2} < \left(\frac{1}{2}\right)^3 a_{n-1}$$

$$\therefore \quad a_{n+m} < \left(\frac{1}{2}\right)^{m+1} a_{n-1}$$

As $m \to \infty$, $a_{n+m} \to 0 \cdot a_{n-1} = 0$

$$\therefore \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Def: RECURSIVELY DEFINED SEQUENCE:

THE NEXT TERM IS CREATED FROM PREVIOUS TERMS.

1. VALUE(S) OF INITIAL TERM(S) GIVEN. (INITIAL VALUES)
2. RECURSION FORMULA (RULE) IS GIVEN FOR CALCULATING LATER TERMS FROM TERMS THAT PRECEDE IT.

e.g. * $\boxed{a_1 = 1, \ a_2 = 1 \quad \text{For } n \geq 3, \ \boxed{a_n = a_{n-2} + a_{n-1}}}$

$n=3$: $a_3 = a_1 + a_2$

$n=4$: $a_4 = a_2 + a_3$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	
FIBONACCI SEQUENCE:	1	1	2	3	5	8	13	...

↑

WHAT IS THE 100^{TH} TERM OF THIS SEQUENCE?

$$a_{100} = a_{98} + a_{99} = (a_{96} + a_{97}) + (a_{97} + a_{98}) = \ldots$$

BUT WHAT ARE THESE?

DRAWBACK OF RECURSIVELY DEFINED SEQUENCES:

DIFFICULT TO CALCULATE a_n WITHOUT CALCULATING ALL TERMS $a_1, \ldots, a_n$.

Def: GIVEN A SEQUENCE a_n, AND 2 REAL NUMBERS b, B.

- B IS AN UPPER BOUND FOR THE SEQUENCE $a_1, a_2, \ldots$ IF

$$a_n \leq B \quad \text{FOR ALL } n.$$

- b IS A LOWER BOUND FOR THE SEQUENCE $a_1, a_2, \ldots$ IF

$$a_n \geq b \quad \text{FOR ALL } n.$$

e.g. SEQUENCE : 3, -3, 3, -3, 3, -3, ...

3 IS AN UPPER BOUND

4, 5, $\sqrt{10}$, 1032, ARE ALL UPPER BOUNDS

ANY # BELOW 3 IS NOT AN UPPER BOUND,

SO WE SAY 3 IS THE LEAST UPPER BOUND.

Def: SUPPOSE B IS AN UPPER BOUND FOR SEQ. a_n.
IF ALL #'S SMALLER THAN B ARE NOT
UPPER BOUNDS FOR a_n, THEN B IS THE
LEAST UPPER BOUND

LOWER BOUND

-2, -2.5,

-1.5,

-1 G.L.B.

-.999 NOT L.B.

e.g. $-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots$

EVERY POSITIVE # IS AN UPPER BOUND.

0 IS THE LEAST UPPER BOUND.

Def: SUPPOSE b IS A LOWER BOUND FOR A SEQ a_n.
IF ALL #'S GREATER THAN b ARE NOT
LOWER BOUNDS FOR a_n, THEN b IS
THE GREATEST LOWER BOUND.

3, -3, 3, -3, ...

-4 L.B. -3 G.L.B.

Def: A SEQUENCE IS <u>NON-DECREASING</u> IF $a_n \le a_{n+1}$.

A SEQUENCE IS <u>NON-INCREASING</u> IF $a_n \ge a_{n+1}$.

<u>MONOTONIC</u> MEANS <u>EITHER</u> NON-INCREASING

OR NON-DECREASING.

e.g. $a_n: 1, 2, 3, 4, 5, \ldots$ NON-DECR.

$b_n: 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ NON-INCR.

BOTH MONOTONIC SEQUENCES.

e.g. $a_n: 1, 2, 2, 2, 3, 3, 4, 9, 9, 12, \ldots$

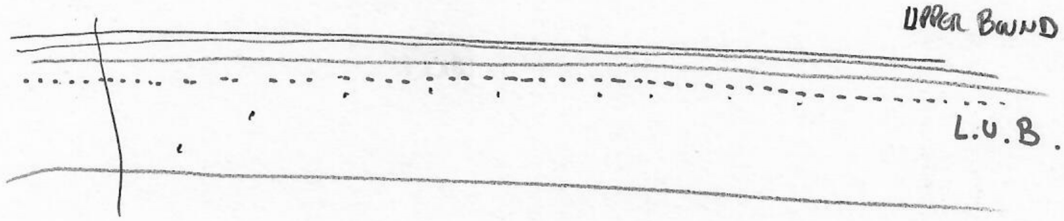
NON-DECREASING. (MONOTONIC)

$b_n: 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{9}, \frac{1}{9}, \frac{1}{12}$

NON-INCREASING (MONOTONIC)

FACTS: EVERY SEQUENCE WITH AN UPPER BOUND HAS A LEAST UPPER BOUND.

EVERY SEQUENCE WITH A LOWER BOUND HAS A GREATEST LOWER BOUND.

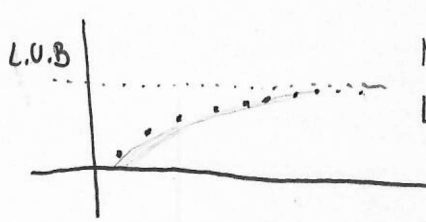


THM. IF A SEQUENCE a_n IS BOUNDED

Def: HAS AN UPPER BOUND (HAS L.U.B.)
HAS A LOWER BOUND (HAS G.L.B.)
(BOTH)

AND THE SEQUENCE IS MONOTONIC

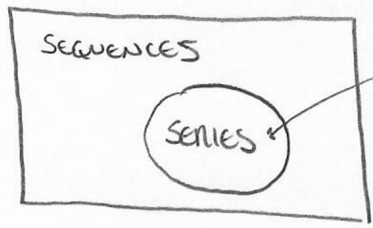
THEN THE SEQUENCE CONVERGES.



NON-DECR.
L.U.B. $\Rightarrow$ SEQ. CONVERGES TO L.U.B.

NON-INCR. $\Rightarrow$ SEQ CONVERGES TO G.L.B.

§10.2 INFINITE SERIES



SPECIAL TYPE OF SEQUENCE.

GIVEN A SEQUENCE $a_1, a_2, a_3, \ldots$

THERE IS A RELATED SEQUENCE, CALLED A SERIES:

$S_1 = a_1$ — 1^{st} PARTIAL SUM

$S_2 = a_1 + a_2$ — 2^{nd} PARTIAL SUM

$S_3 = a_1 + a_2 + a_3$ — 3^{rd} PARTIAL SUM

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$ — n^{TH} PARTIAL SUM

$\hookrightarrow \quad S_n = \sum_{i=1}^{n} a_i$

Def: IF SEQUENCE OF PARTIAL SUMS $S_n = \sum_{i=1}^{n} a_i$ CONVERGES TO A LIMIT L, THEN WE SAY THE SERIES $\sum_{i=1}^{\infty} a_i$ CONVERGES & THAT ITS SUM IS L. IF THE SEQUENCE OF PARTIAL SUMS DIVERGES, THEN WAY SAY THE SERIES DIVERGES.

e.g. GEOMETRIC SEQUENCE : $\boxed{a_n = ar^{n-1}}$ EXPLICIT

$$\begin{aligned} a_1 &= a \\ a_2 &= ar \\ a_3 &= ar^2 \\ a_4 &= ar^3 \\ &\vdots \end{aligned} \qquad \times r$$

RECURSIVE: $\boxed{\begin{aligned} a_1 &= a \\ a_n &= r a_{n-1} \end{aligned}}$

<u>GEOMETRIC SERIES</u> : $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$ SUM OF ALL TERMS IN A GEOMETRIC SERIES.

e.g. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^{n-1} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n} = \sum_{n=5}^{\infty} \left(\frac{1}{4}\right)^{n-4}$

(R E IN DE X IN G)

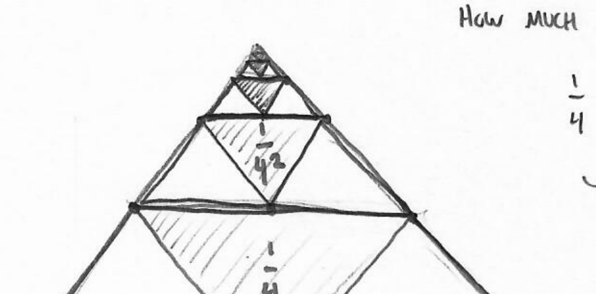
$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \frac{1}{4^5} + \dots$$

THIS GEOMETRIC SERIES CONVERGES TO $\frac{1}{3}$.

EQUILATERAL TRIANGLE

TOTAL AREA 1

HOW MUCH AREA IS SHADED?



$$\underbrace{\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots}_{} \leq 1$$

INCREASING SEQUENCE OF PARTIAL SUMS $\Rightarrow$ CONVERGE

$\frac{1}{3}$

TOTAL AREA OF ALL  = 1

EXACTLY $\frac{1}{3}$ OF ALL  IS SHADED