§10.2 INFINITE SERIES

SEQUENCE: ORDERED LIST OF TERMS $a_1, a_2, a_3, \ldots$

LIMIT OF A SEQUENCE $\lim_{n \to \infty} a_n = L$

$$\left(|a_n - L| \to 0 \text{ AS } n \to \infty \right)$$

SERIES (INFINITE SERIES) SUM OF ALL TERMS OF A SEQUENCE

$$a_1 + a_2 + a_3 + \ldots$$

THE SERIES (SUM) IS THE LIMIT OF THE SEQUENCE OF PARTIAL SUMS

$$\sum_{n=1}^{\infty} a_n = \lim_{m \to \infty} S_m$$

WHERE S_m IS THE m^{TH} PARTIAL SUM

$$S_m = \sum_{n=1}^{m} a_m = a_1 + a_2 + \ldots + a_m$$

GEOMETRIC SEQUENCE: $a_n = ar^{n-1}$ $\quad a \neq 0, \ r \neq 0$

GEOMETRIC SERIES $\sum_{n=1}^{\infty} ar^{(n-1)} \to \underset{n=1}{3} - \underset{n=2}{\frac{3}{2}} + \underset{n=3}{\frac{3}{4}} - \frac{3}{8} + \frac{3}{16} - \ldots$

eg. $a = 3, \ r = -\frac{1}{2}$

$$\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{3} \right)$$

IF $r = 1$; $\sum_{n=1}^{\infty} ar^{n-1}$ D.N.E. WHEN $a \neq 0$.

GEOMETRIC SERIES:

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{m \to \infty} S_m = \lim_{m \to \infty} \sum_{n=1}^{m} a_n$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$\left.\begin{aligned} S_n &= a \overset{\times r}{+} ar \overset{\times r}{+} ar^2 + \dots + ar^{n-1} \\ -\left(rS_n \right. &= \left. ar + ar^2 + ar^3 + \dots + ar^n \right) \end{aligned}\right\} \text{TELESCOPING SUM}$$

$$S_n - rS_n = a + ar^n$$

$$S_n(1-r) = a(1 + ar^n)$$

$$S_n = \frac{a(1+r^n)}{1-r}$$

RECALL:

IF $|r| < 1$: $\lim_{n \to \infty} r^n = 0$

IF $|r| > 1$: $\lim_{n \to \infty} r^n$ D.N.E.

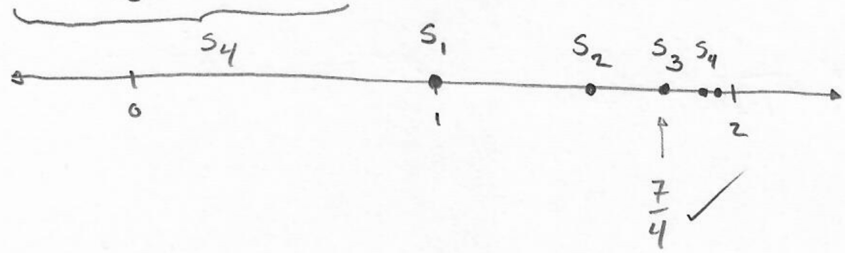
$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1+r^n)}{1-r}$$

$$\therefore \sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} \text{ (CONVERGES)} & \text{IF } |r| < 1 \\ \text{DIVERGES} & \text{IF } |r| \geq 1 \end{cases}$$

$a = 1^{ST}$ TERM $\neq 0$

NOTE: GEOMETRIC SERIES ARE REALLY THE ONLY SERIES THAT WE WILL BE ABLE TO CALCULATE EXACTLY. ALL OTHER SERIES WE WILL ONLY BE ABLE SAY WHETHER THEY CONVERGE OR DIVERGE.

e.g. $$\underbrace{\underbrace{\overbrace{\overbrace{1}^{s_1} + \frac{1}{2}}^{s_2} + \frac{1}{4}}_{s_3} + \frac{1}{8}}_{s_4} + \frac{1}{16} + \dots$$



$$S_n = \frac{a(1-r^n)}{1-r} \xrightarrow{n=3} \frac{1\left(1-\left(\frac{1}{2}\right)^3\right)}{1-\frac{1}{2}} = \frac{\frac{7}{8}}{\frac{1}{2}} = \frac{7}{4}$$

$$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} 1\left(\frac{1}{2}\right)^{n-1} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \boxed{2}$$

$a=1 \quad r=\frac{1}{2}$

ex. EXPRESS THE REPEATING DECIMAL 3.138138138... AS A $\overbrace{\text{RATIONAL NUMBER}}^{\text{FRACTION}}$

$$3.138138138\ldots = 3 + \frac{138}{1000} + \frac{138}{1{,}000{,}000} + \frac{138}{1{,}000{,}000{,}000}$$

$$= 3 + \underbrace{138 \cdot \frac{1}{1000} \xrightarrow{\times \frac{1}{1000}} + 138\left(\frac{1}{1000}\right)^2 + 138\left(\frac{1}{1000}\right)^3 + \dots}_{\text{GEOMETRIC } \cancel{\text{SEQUENCE}} \text{ SERIES}}$$

$$\frac{a}{1-r} = \sum_{n=1}^{\infty} ar^{n-1}$$

$a = 1^{st}$ TERM $= \frac{138}{1000}$

$r =$ RATIO FROM ONE TERM TO NEXT

$= \frac{1}{1000}$

$$= 3 + \sum_{n=1}^{\infty} \frac{138}{1000}\left(\frac{1}{1000}\right)^{n-1} = 3 + \frac{\frac{138}{1000}}{1-\frac{1}{1000}} = 3 + \frac{138}{\cancel{1000}} \cdot \frac{\cancel{1000}}{999}$$

$$= 3 + \frac{138}{\;} = \frac{3135}{\;}$$

THM (DIVERGENCE TEST)

IF $\sum_{n=1}^{\infty} a_n$ CONVERGES, THEN $\lim_{n\to\infty} a_n = 0$.

SUMMING UP ∞ TERMS TO GET A FINITE #.

IF THE TERMS DIDN'T GET SMALL, THEN THE SUM WOULD DIVERGE.

EQUIVALENTLY

$$\boxed{\text{IF } \lim_{n\to\infty} a_n \neq 0, \text{ THEN } \sum_{n=1}^{\infty} a_n \text{ DIVERGES.}}$$

e.g. $\sum_{n=1}^{\infty} \frac{(n+1)(n-1)}{2n^2}$ DOES THIS SERIES CONVERGE OR DIVERGE?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{2n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}}{2\cancel{n^2}} - \frac{1}{2n^2}$$

$$= \frac{1}{2} \neq 0$$ $\therefore$ SERIES DIVERGES BY DIVERGENCE TEST

THM: IF $\sum_{n=1}^{\infty} a_n = A$ AND $\sum_{n=1}^{\infty} b_n = B$ & $c \in \mathbb{R}$ THEN

1. $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$
2. $\sum_{n=1}^{\infty} c a_n = cA$

SERIES ARE LIMITS (OF PARTIAL SUMS) SO LIMIT RULES APPLY.

GEOM. SERIES

ex. EVALUATE THE SERIES $\displaystyle\sum_{n=1}^{\infty} \frac{2^n - 3\cdot 2^{2n-2}}{8^{n-1}}$ $\quad (2^2)^{(n-1)} = 4^{n-1}$

$$= \sum_{n=1}^{\infty} \frac{2^n}{8^{n-1}} - \sum_{n=1}^{\infty} \frac{3\cdot 2^{2n-2}}{8^{n-1}}$$

GEOMETRIC SERIES

$$\left(\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \right)$$

$$\left(\begin{array}{l} a = 1^{ST} \text{ TERM} \\ |r| < 1 \text{ *OTHERWISE DIVERGES*} \end{array} \right)$$

$$= \sum_{n=1}^{\infty} 2\left(\frac{2}{8}\right)^{n-1} - \sum_{n=1}^{\infty} 3\left(\frac{4}{8}\right)^{n-1}$$

$a = 2$ $\qquad a = 3$

$r = \frac{1}{4}$ $\qquad r = \frac{1}{2}$

$$= \frac{2}{1-\frac{1}{4}} - \frac{3}{1-\frac{1}{2}} = \frac{2}{\frac{3}{4}} - \frac{3}{\frac{1}{2}}$$

$$= 2\cdot\frac{4}{3} - 3\cdot 2 = \frac{8}{3} - 6$$

$$= \boxed{-\frac{10}{3}}$$

ex. FOR WHAT VALUES OF X DOES $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2x-4}{3}\right)^{\boxed{n}}$ CONVERGE

REINDEXING

$$= \sum_{n=1}^{\infty} 1\left(\frac{2x-4}{3}\right)^{\boxed{n-1}}$$

GEOMETRIC SERIES $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

WITH $a = 1$, $r = \frac{2x-4}{3}$

* WHEN $|r| < 1$

* $\left|\frac{2x-4}{3}\right| < 1 \longrightarrow -1 < \frac{2x-4}{3} < 1$

$-3 < 2x-4 < 3 \longrightarrow 1 < 2x < 7$

$$\boxed{\frac{1}{2} < x < \frac{7}{2}}$$

§10.3 THE INTEGRAL TEST

THM (INTEGRAL TEST)

SUPPOSE a_n IS A SEQUENCE OF POSITIVE TERMS, AND THERE EXISTS A FUNCTION $f(x)$ SUCH THAT

EVENTUALLY

1. $f(n) = a_n$ FOR POSITIVE INTEGERS $n \geq N$, WHERE N IS SOME POSITIVE INTEGER.

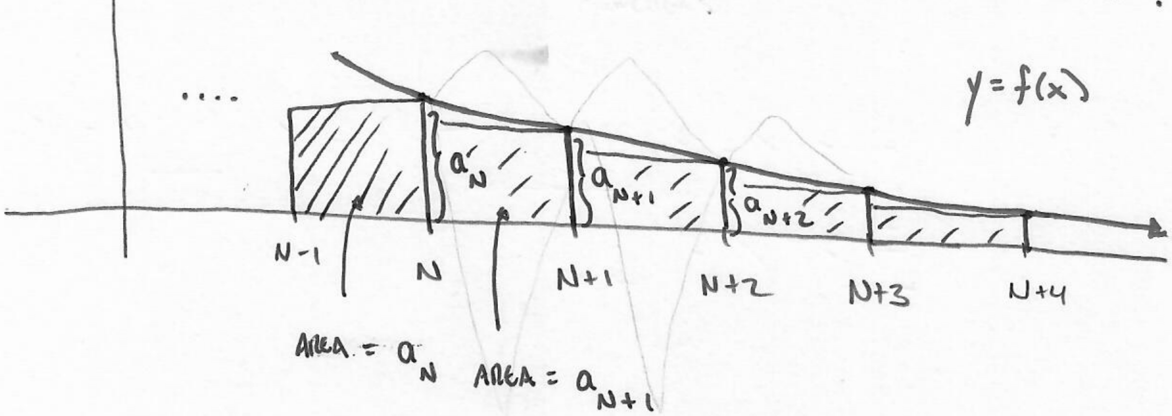
2. $f(x)$ IS POSITIVE & DECREASING FOR ALL REAL NUMBERS $x \geq N$ (EVENTUALLY).

THEN THE SERIES $\sum_{n=1}^{\infty} a_n$ AND THE INTEGRAL (IMPROPER) $\int_N^{\infty} f(x)\,dx$ EITHER BOTH CONVERGE OR BOTH DIVERGE.

CONVERGES OR DIVERGES? $\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + \dots + a_N}_{\text{FINITE SUM}} + \underbrace{a_{N+1} + a_{N+2} + \dots}_{\text{IS THIS FINITE ?}}$

TAIL

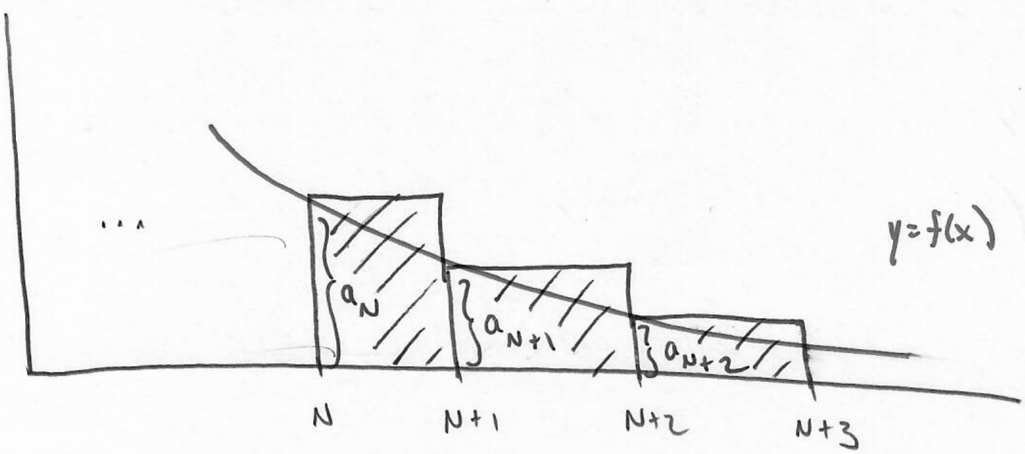
IDEA:



$$\sum_{n=N}^{\infty} a_N = \boxed{} \leq \int_N^{\infty} f(x)\,dx$$

SO IF $\int_N^{\infty} f(x)\,dx$ CONVERGES, THEN SO DOES $\sum_{n=N}^{\infty} a_n$

THEN SO DOES $\underbrace{\sum_{n=1}^{N-1} a_n}_{\text{FINITE}} + \sum_{n=N}^{\infty} a_n = \sum_{n=1}^{\infty} a_n$



$$\sum_{n=N}^{\infty} a_n = \text{AREA } \boxed{} \geq \int_N^{\infty} f(x)\,dx$$

$\therefore$ IF $\int_N^{\infty} f(x)\,dx$ DIVERGES THEN $\sum_{n=1}^{\infty} a_n$ ALSO DIVERGES.

ex. (P-SERIES TEST) $\sum_{n=1}^{\infty} a_n$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$(p>0)$

$f(x) = x^{-p}$

APPLY INTEGRAL TEST :

$f(x) = \frac{1}{x^p}$

1. $f(n) = \frac{1}{n^p} = a_n \quad \forall n \geq 1$

2. $f(x)$ IS DECREASING & POSITIVE

$f(x) = \frac{1}{x^p} > 0$ WHEN $x \geq 1$ ✓

$f'(x) = -p\,x^{-p-1}$ DECREASING ✓

(WHEN $p>0$)

∴ BY INTEGRAL TEST $\sum_{n=1}^{\infty} \frac{1}{n^p}$ HAS THE SAME BEHAVIOR (CONV. OR DIV.)

AS IMPROPER INT $\int_1^{\infty} \frac{1}{x^p}\,dx$.

BY p-TEST, THE IMP INT. CONVERGES WHEN $p>1$

& DIVERGES WHEN $p \leq 1$

∴ (P-SERIES TEST) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{CONVERGES IF } p>1 \\ \text{DIVERGES IF } p \leq 1 \end{cases}$

ex. CONVERGE OR DIVERGE? $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$ INTEGRAL TEST

$$f(x) = \frac{x^2}{e^{x/3}} = x^2 e^{-x/3}$$

1. $f(n) = a_n$ FOR ALL $n \geq 1$ ✓
2. $f(x) > 0$ FOR ALL $x \geq 1$ ✓

IS $f(x)$ DECREASING? ON WHAT INTERVALS?

$$f'(x) = 2x e^{-x/3} - \frac{1}{3} x^2 e^{-x/3}$$

$$f'(x) = x e^{-x/3} \left(2 - \frac{1}{3}x\right) = 0$$

		0		6	
$f'(x)$	$-$	0	$+$	0	$-$
$x e^{-x/3}$	$-$		$+$		$+$
$2 - \frac{1}{3}x$	$+$		$+$		$-$

$\therefore$ $f(x)$ IS DECREASING WHEN $x \geq 6$ ✓

$\therefore$ BY INTEGRAL TEST: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$ HAS THE SAME BEHAVIOR AS

$$\int_6^{\infty} x^2 e^{-x/3}\, dx$$

INTEGRATION BY PARTS REPEATED.

(TABULAR METHOD)   L.I.A.T.E.   u

	u		dv	
u =	x^2	$+$	$e^{-x/3}$	$= dv$
u = du =	$2x$	$-$	$-3e^{-x/3}$	$= v$ dv
	2	$+$	$9e^{-x/3}$	
	0		$-27e^{-x/3}$	

$$\int_6^\infty x^2 e^{-x/3}\,dx = -3x^2 e^{-x/3}\Big|_6^\infty + \int 6x e^{-x/3}\,dx$$

$=$

$$\int_6^\infty x e^{-x/3}\,dx = \lim_{t\to\infty}\left[-3x^2e^{-x/3} - 18xe^{-x/3} - 54e^{-x/3}\right]_6^t$$

$$= \lim_{t\to\infty} -3t^2e^{-t/3} - 18te^{-t/3} - 54e^{-t/3} \quad \text{L'HÔ} \quad 0$$

$$\underbrace{+\,3(6)^2e^{-2} + 18(6)e^{-2} + 54e^{-2}}$$

IMP. INT.   CONVERGES

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$ CONVERGES BY INT. TEST.

QUIZ TUE. 6/23

§ 8.8, 10.1, 10.2, 10.3

- EVALUATE
- DIRECT / LIMIT COMPARISON THM.