§10.5 ABS. CONV. & RATIO TEST, ROOT TEST

ROOT TEST: SERIES $\sum a_n$ WITH

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} .$$

THEN
$$\sum a_n \begin{cases} \text{CONVERGES ABSOLUTELY IF } \rho < 1 \\ \text{DIVERGES IF } \rho > 1 \\ \text{INCONCLUSIVE IF } \rho = 1 . \end{cases}$$

PROOF: WHEN n IS LARGE (SAY $n \geq N$) WE HAVE

$$\sqrt[n]{|a_n|} = |a_n|^{1/n} \approx \rho$$

$$\Rightarrow \boxed{|a_n| \approx \rho^n}$$

NOTE: IF $\rho > 1$

THEN $|a_n| \not\to 0$

$\left(\lim_{n\to\infty} |a_n| = \infty \right)$

$\therefore$ SERIES DIVERGES BY DIV. TEST. ✓

IF $\rho < 1$:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$$

$$\approx \underbrace{\sum_{n=1}^{N-1} |a_n|}_{\text{FINITE}} + \sum_{n=N}^{\infty} \rho^n$$

GEO. SERIES CONVERGES SINCE $\rho < 1$ ✓

IF $\rho = 1$: INCONCLUSIVE : e.g. $\sum \frac{1}{n}$ DIVERGES $\qquad \sum \frac{1}{n^2}$ CONVERGES

$\lim \sqrt[n]{\frac{1}{n}} = 1 \qquad \lim \sqrt[n]{\frac{1}{}} = 1$

RECALL: •) $$\lim_{n\to\infty} \left(n^a\right)^{1/n} = e^{\ln\left(\lim\limits_{n\to\infty} n^{a/n}\right)}$$

$$= e^{\lim\limits_{n\to\infty} \frac{a\ln(n)}{n}} \overset{\text{L'HO}}{=} e^{\lim\limits_{n\to\infty} \frac{a}{n}} = e^0 = 1$$

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ex. $$\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$$ CONV. OR DIV. ? $$a_n = \left(\frac{4}{3n}\right)^{\boxed{n}}$$

ROOT TEST: $$\rho = \lim_{n\to\infty} |a_n|^{\boxed{1/n}} = \lim_{n\to\infty} \frac{4}{3n} = 0$$

∴ SERIES CONVERGES (ABSOLUTELY) BY ROOT TEST.

ex. $$\sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{n-1} \underset{\text{REINDEX}}{=} \sum_{n=0}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n+1}\right)\right)^n$$

ROOT TEST: $$\underline{\underline{\rho}} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\left(\cancel{-}\ln\left(e^2 + \frac{1}{n+1}\right)\right)^{\cancel{n}}\right|^{\cancel{1/n}}$$

$$= \lim_{n\to\infty} \ln\left(e^2 + \underset{\to 0}{\boxed{\frac{1}{n+1}}}\right) = \ln(e^2 + 0) = \underline{\underline{2}}$$

SINCE $\rho > 1$, THE SERIES DIVERGES. BY ROOT TEST.

FACTORIALS ⟹ CONSIDER RATIO TEST.

ex. CONV. OR DIV.? $\sum_{n=1}^{\infty} \frac{n!\,\ln(n)}{n\,(n+2)!} = \sum \frac{\cancel{n!}\,\ln(n)}{n\,\underbrace{(n+2)(n+1)\cancel{n!}}_{(n+2)!}}$

WHAT TEST SHOULD WE USE?

$\sum_{n=1}^{\infty} \frac{\ln(n)}{n\,(n+2)(n+1)}$

DIRECT COMP. TEST

RATIO TEST. INCONCLUSIVE

RATIO TEST $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\ln(n+1)}{\cancel{(n+1)}\cancel{(n+2)}(n+3)} \cdot \frac{n\cancel{(n+1)}\cancel{(n+2)}}{\ln(n)} \right|$

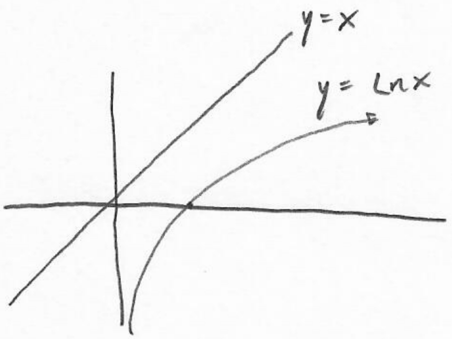
$= \lim_{n\to\infty} \left(\frac{n}{n+3}\right) \cdot \left(\frac{\ln(n+1)}{\ln(n)}\right)$ L'HO ×2 $=$

$\to 1 \qquad \to 1$

$= \left(\lim_{n\to\infty} \frac{1}{1+0}\right)\left(\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}}\right) = 1 \cdot 1 = 1.$

$0 \le a_n = \frac{\ln(n)}{n\,(n+2)(n+1)} \le \boxed{\frac{\cancel{n}}{\cancel{n}\,(n+2)(n+1)}}$

$\le \frac{1}{n^2}$



SINCE $\sum \frac{1}{n^2}$ CONV. BY p-TEST $(p=2>1)$,

ORIGINAL SERIES ALSO CONVERGES BY DIRECT COMPARISON TEST.

§ 10.6 ALTERNATING SERIES & CONDITIONAL CONVERGENCE

Def: A SERIES $\sum_{n=1}^{\infty} a_n$ IS ALTERNATING IF IT HAS THE FORM

$$\pm \sum_{n=1}^{\infty} (-1)^{n+1} \boxed{u_n}, \qquad u_n > 0$$

1st TERM COULD BE + OR -

$u_n = |a_n|$

+1, -1, +1, -1 : SIGN ALTERNATES.

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

(THIS IS AN ALTERNATING SERIES!

$$+ \sum_{n=1}^{\infty} (-1)^{n+1} \boxed{\frac{1}{n}} \qquad u_n = \frac{1}{n} \left(= |a_n| \right)$$

ALTERNATING SERIES TEST:

$$\underbrace{\underbrace{\overbrace{\overset{s_1}{\widetilde{u_1}} - u_2}^{s_2} + u_3}_{s_3} - u_4}_{s_4} + u_5 - u_6 + \dots$$

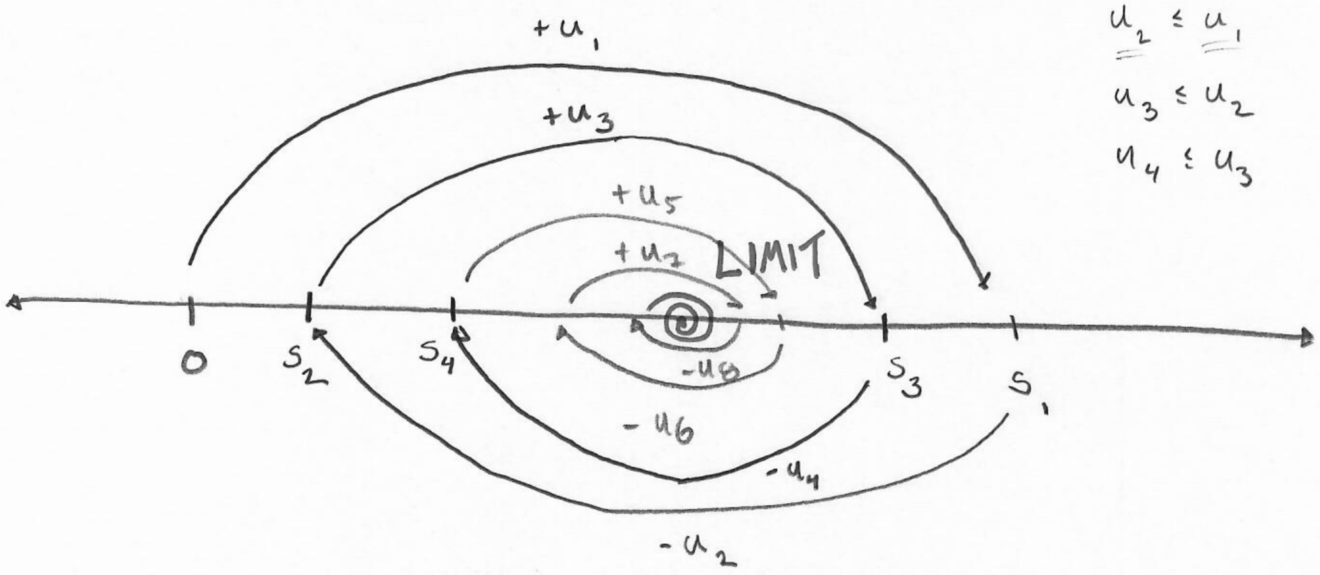
AN ALTERNATING SERIES $\pm \sum_{n=1}^{\infty} (-1)^{n+1} u_n$, $u_n > 0$

CONVERGES IF 1. EVENTUALLY $u_{n+1} \le u_n$

(FOR ALL $n \ge N$, WE HAVE $u_{n+1} \le u_n$)

2. $\lim_{n\to\infty} u_n = 0$.

PROOF: LETS ASSUME $u_{n+1} \le u_n$ FOR ALL n, $\lim_{n\to\infty} u_n = 0$.



$u_2 \le u_1$

$u_3 \le u_2$

$u_4 \le u_3$

$$\left(\sum a_n = \lim_{n\to\infty} \overset{\substack{n^{TH} \text{ PARTIAL SUM} \\ \downarrow}}{S_n} \quad , \quad S_n = \sum_{i=1}^{n} a_n \right)$$

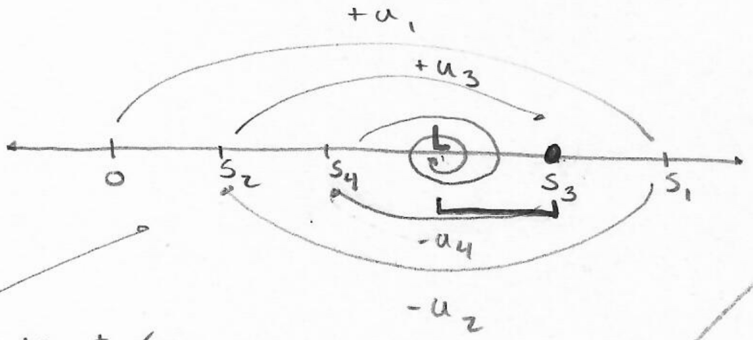
SERIES CONVERGES $\Longleftrightarrow$ SEQUENCE OF PARTIAL SUMS CONVERGE

ALTERNATING SERIES ESTIMATION THM.

SUPPOSE AN ALTERNATING SERIES CONVERGES.

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = L$$

THEN WE CAN APPROXIMATE $L \approx S_n = u_1 - u_2 + u_3 - \dots \boxed{+ (-1)^{n+1} u_n}$



EVERY TIME WE ± THE NEXT TERM u_n, THE PARTIAL SUM "JUMPS OVER" THE LIMIT.

$\therefore \ |S_n - L|$

$$|s_3 - L| \leq u_4$$

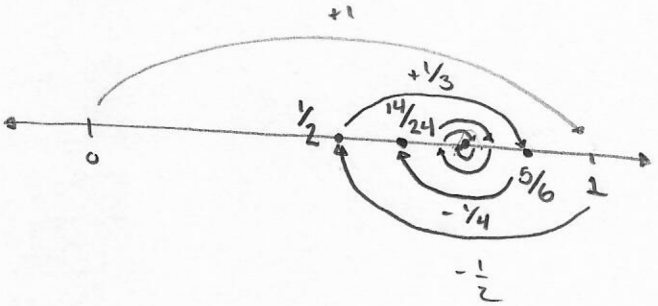
NOTE:
- IF THE LAST TERM IN PARTIAL SUM IS POSITIVE, ~~THEN~~ THEN THIS IS AN OVERESTIMATE
- IF THE LAST TERM IN PARTIAL SUM IS NEGATIVE, THEN THIS IS AN UNDERESTIMATE.

AND THE ERROR $|S_n - L| \leq u_{n+1}$.

"ERROR BOUND"

ex. "ALTERNATING HARMONIC SERIES" $\overbrace{\overbrace{\overbrace{1} - \frac{1}{2}} + \frac{1}{3}} - \frac{1}{4} + \frac{1}{5} - \dots$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$



APPROXIMATE $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ BY $\boxed{S_{30} = .67676}$

OVERESTIMATE OR UNDERESTIMATE?

n ODD $\Rightarrow$ S_n OVEREST.

n EVEN $\Rightarrow$ S_n UNDEREST.

S_{30} IS UNDERESTIMATE

MAXIMUM ERROR? $|S_n - L| \leq u_{n+1}$

ERROR $\leq u_{31} = \frac{1}{31} \approx .032258$

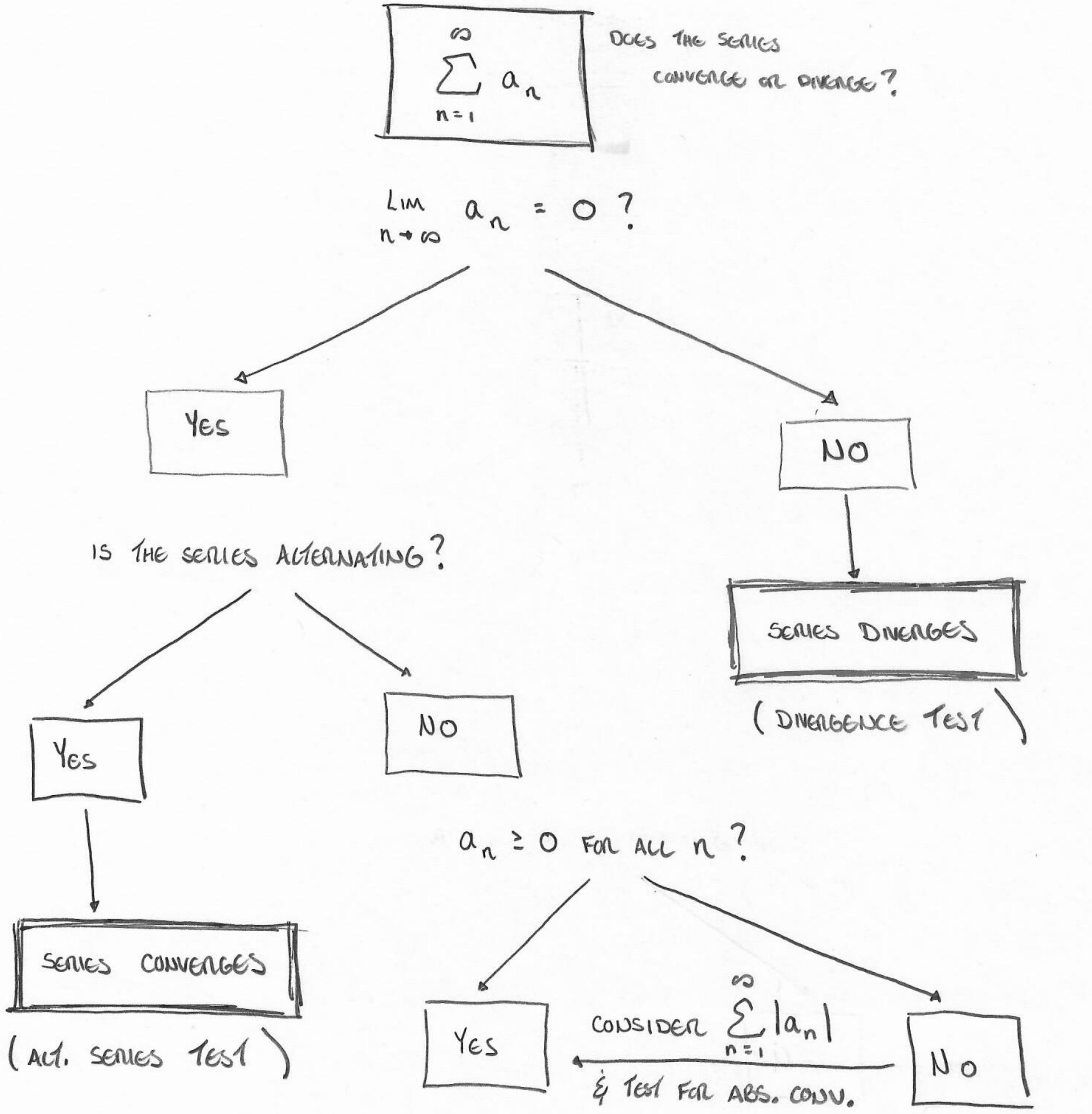
ALT. HARMONIC SERIES

SO: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ CONVERGES BUT DOES NOT CONVERGE ABSOLUTELY

$\left(\sum_{n=1}^{\infty} \frac{1}{n} \text{ DIVERGES} \right)$

HARMONIC SERIES

Def: A SERIES THAT CONVERGES BUT DOES NOT ABSOLUTELY CONVERGE IS SAID TO CONVERGE CONDITIONALLY.



- INTEGRAL TEST
- DIRECT COMPARISON TEST
- LIMIT COMPARISON TEST
- RATIO TEST $(n!)$
- ROOT TEST $(\ )^n$

REMEMBER:
ABS. CONV.
IMPLIES CONV.

ex. CONVERGE OR DIVERGE? $$\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$$

$$\left(\lim_{n\to\infty} |a_n| = 0 \iff \lim_{n\to\infty} a_n = 0 \right)$$

$$\lim_{n\to\infty} \frac{10^n}{(n+1)!} = 0$$

ALT. SERIES WITH TERMS SHRINKING TO 0

$$\Downarrow$$

$$\boxed{\text{CONVERGES}}$$

BY ALT. SERIES TEST.

Note: $$a_n = \frac{a^n}{n!} = \frac{a}{n} \cdot \frac{a^{n-1}}{(n-1)!}$$

$$a_n = \frac{a}{n} \cdot a_{n-1}$$

∴ WHEN $n > a$ WE HAVE

$$a_n = \frac{a}{n} a_{n-1}$$

$$a_{n+k} \leq \left(\frac{a}{n}\right)^k a_{n-1}$$

$$\lim_{n\to\infty} \frac{a^n}{n!} \leq \underbrace{\lim_{k\to\infty} \left(\frac{a}{n}\right)^k a_{n-1}}_{=0}$$

EX. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^{n}$ CONV. OR DIV. ?

CONSIDER $\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left(\frac{n}{10}\right)^{n} = \infty \neq 0$

$n = 10 \quad |a_{10}| = 1$

$n = 100 \quad |a_{100}| = 10^{100}$

$n = 1000 \quad |a_{1000}| = 100^{1000}$

$\displaystyle\lim_{n\to\infty} a_n \neq 0$, SO SERIES <u>DIVERGES</u> BY DIVERGENCE TEST.

EX. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} \ln\left(1+\frac{1}{n}\right)$

↑

ALTERNATING, SO IT CONVERGES IF $|a_n| \to 0$

$\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln(1) = 0$ ✓

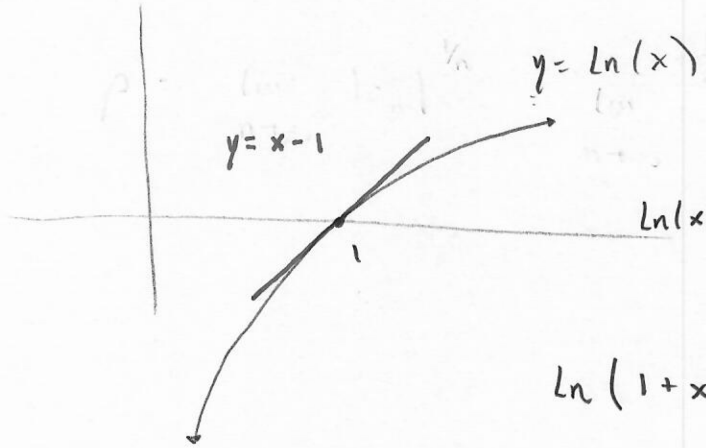
∴ THE SERIES <u>CONVERGES.</u>

DOES THE SERIES CONVERGE <u>ABSOLUTELY</u> (OR <u>CONDITIONALLY</u>)?

ABSOLUTE CONVERGENCE ?

YES OR NO ?

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$$



$\ln(x) \approx x - 1$ FOR X CLOSE TO 1

$\ln(1+x) \approx x$ FOR X CLOSE TO 0

$\ln\left(1+\frac{1}{n}\right) \approx \boxed{\frac{1}{n}}$ FOR LARGE n

DIVERGES

ALSO DIVERGES.