MIDTERM EXAM TOMORROW 8:30 - 10:10 AM (FULL CLASS)

ZOOM MEETING OPTIONAL FOR QUESTIONS

PEARSON

OFFICE HOURS

MON, WED 1:30 - 3:30 PM

§10.7 POWER SERIES

n^{TH} COEFFICIENT

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{POWER SERIES ABOUT } x=0$$

$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

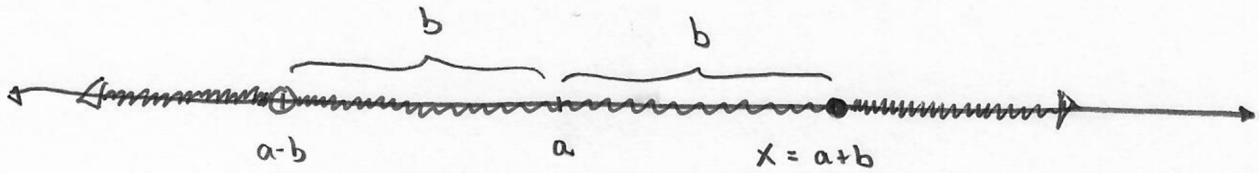
HOR. SHIFT

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{POWER SERIES ABOUT } x=a$$

a IS THE CENTER OF THE POWER SERIES

POWER SERIES CONVERGENCE THEOREM (GENERALIZATION)

IF THE POWER SERIES $\sum_{n=0}^{\infty} c_n (x-a)^n$ CONVERGES (DIVERGES) AT $x-a = b \neq 0$, THEN IT CONVERGES (DIVERGES) FOR ALL x SUCH THAT $|x-a| < |b|$ $\left(|x-a| > |b|\right)$.

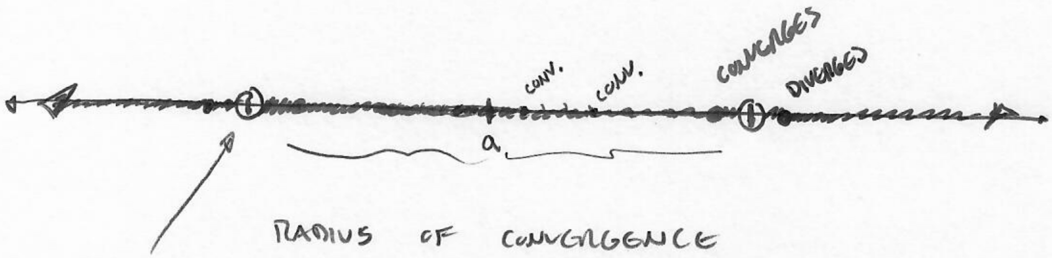


IF $\sum c_n (x-a)^n$ CONVERGES HERE — CONVERGES "CLOSER TO a"

IF $\sum c_n (x-a)^n$ DIVERGES HERE — DIVERGES "FARTHER FROM a"

THIS IMPLIES: WHEN ASKED FOR WHAT VALUES DOES A POWER SERIES CONVERGE?

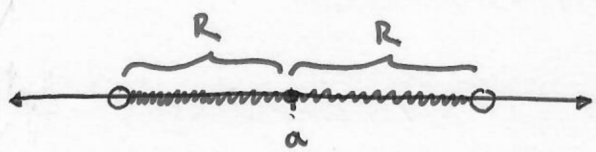
$$\sum c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$



RADIUS OF CONVERGENCE

CONVERGENCE & DIVERGENCE HERE MUST BE DETERMINED SEPARATELY.

- IT'S GOING TO BE AN INTERVAL CENTERED AT $x=a$.



R IS CALLED THE RADIUS OF CONVERGENCE.

3 CASES FOR R:

ABS. VAL. OF DIFFERENCE = DISTANCE BETWEEN #'S.

1. R IS POSITIVE, FINITE.

   $\Rightarrow$ SERIES CONVERGES FOR $|x-a| < R$

   DIVERGES FOR $|x-a| > R$

   AT ENDPOINTS, WHERE $x - a = R$, WE MUST CHECK FOR CONVERGENCE / DIVERGENCE EXPLICITLY.

2. $R = \infty$

   $\Rightarrow$ SERIES CONVERGES FOR ALL x

3. $R = 0$ THEN SERIES <u>ONLY</u> CONVERGES AT $x = a$,

HOW TO FIND THE <u>INTERVAL</u> OF CONVERGENCE FOR A POWER SERIES:

CASE 1: $(x-R, x+R)$

$(x-R, x+R]$

$[x-R, x+R)$

$[x-R, x+R]$

CASE 2: $(-\infty, \infty)$

CASE 3: $\{a\} = [a, a]$

STEP 1: USE RATIO / ROOT TEST TO FIND THE RADIUS OF CONVERGENCE.

$$\rho = \begin{cases} \text{RATIO: } \lim_{n\to\infty} \left| \dfrac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| \\ \text{ROOT } \lim_{n\to\infty} \left| c_n (x-a)^n \right|^{1/n}. \end{cases}$$

$$= \begin{cases} \text{RATIO } \left(\lim_{n\to\infty} \left| \dfrac{c_{n+1}}{c_n} \right| \right) |x-a| \\ \text{ROOT } \left(\lim_{n\to\infty} |c_n|^{1/n} \right) |x-a| \end{cases}$$

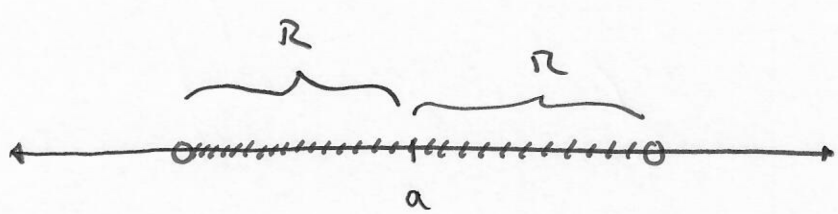
SERIES CONVERGES WHEN $\rho < 1$

DIVERGES WHEN $\rho > 1$

INCONCLUSIVE WHEN $\rho = 1$

SET $\rho < 1$ $\longrightarrow$ EXPRESSION INVOLVING $|x-a| < 1$

$\longrightarrow$ $|x-a| < R$ RADIUS OF CONVERGENCE.



CONVERGES.

IF $R = \infty$ OR $R = 0$

$\Rightarrow$ DONE!

STEP 2: IF $0 < R < \infty$ PLUG IN THESE VALUES.

THEN TEST ENDPOINTS $x = a \pm R$ EXPLICITLY

FOR CONVERGENCE USING

- DIRECT COMP THM
- LIMIT COMP THM
- INTEGRAL TEST
- ALT SERIES TEST

DO NOT USE RATIO TEST / ROOT TEST

BECAUSE YOU WILL GET AN INCONCLUSIVE RESULT.

ex. FIND RADIUS OF CONVERGENCE & ALL VALUES X S.T. THE POWER SERIES CONVERGES.

$$C_n \quad (x - a)^n$$

$$\sum_{n=0}^{\infty} \underbrace{\frac{n(x+3)^n}{5^n}}_{a_n} = \sum_{n=0}^{\infty} \frac{n}{5^n} \left(x - (-3)\right)^n \qquad \text{center } a = -3$$

$$= 0 + \frac{1}{5}\left(x-(-3)\right) + \frac{2}{25}\left(x-(-3)\right)^2 + \dots$$

STEP 1: RATIO/ROOT TEST:

RATIO TEST: $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \cancel{\lim_{n\to\infty} \left| (n+1)(x+3)^{n+1} \right.}$

$$= \lim_{n\to\infty} \left| \frac{(n+1)(x+3)^{\cancel{n+1}}}{5^{\cancel{n+1}}} \cdot \frac{\cancel{5^n}}{n\cancel{(x+3)^n}} \right|$$

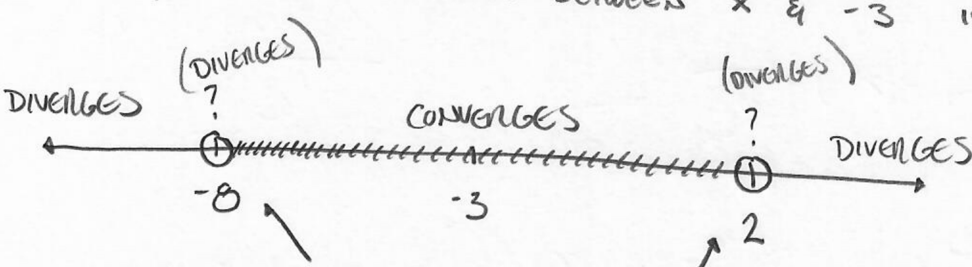
$$= \lim_{n \to \infty} \left| \frac{n+1}{n} \frac{(x+3)}{5} \right|$$

$$= \underbrace{\lim_{n \to \infty} \left| \frac{n+1}{n} \right|}_{1} \cdot \left| \frac{x+3}{5} \right| = \left| \frac{x+3}{5} \right|$$

SERIES CONVERGES WHEN $\rho < 1$, i.e. $\rho = \left| \frac{x+3}{5} \right| < 1$

$|x+3| < 5$ RADIUS OF CONVERGENCE $R = 5$

$|x - (-3)| < 5$ DISTANCE BETWEEN x & -3 IS < 5



TEST ENDPOINTS SEPARATELY

STEP 2: $x = -8 \rightarrow \sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n} \rightarrow \sum_{n=0}^{\infty} \frac{n(-5)^n}{5^n}$

$$= \sum_{n=0}^{\infty} \frac{n(-1)^n \cancel{5^n}}{\cancel{5^n}} = \sum_{n=0}^{\infty} (-1)^n n = 0 - 1 + 2 - 3 + 4 - 5 \cdots$$

DIVERGES

$x = 2$: $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n} \longrightarrow \sum_{n=0}^{\infty} \frac{n\,\cancel{5^n}}{\cancel{5^n}}$ DIVERGES BY DIVERGENCE TEST (TERMS (n) DO NOT $\to 0$)

$\hookrightarrow$ DIVERGES

RADIUS OF CONV. 5

VALUES FOR WHICH SERIES CONVERGES: $(-8, 2)$

ex. FIND RADIUS OF CONVERGENCE AND ALL VALUES x S.T. POWER SERIES CONVERGES.

$$\sum_{n=0}^{\infty} n^n x^n$$

STEP 1: ROOT TEST: $\rho = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |n^n x^n|^{1/n}$

$$= \lim_{n\to\infty} |nx| = \begin{cases} \infty \text{ IF } x \neq 0 \\ 0 \text{ IF } x = 0 \end{cases}$$

ROOT TEST: IF $\rho < 1$ CONVERGES $\quad \rho = 0 < 1$ IF $x = 0$ (CONV.)

IF $\rho > 1$ DIVERGES $\quad \rho = \infty > 1$ IF $x \neq 0$ (DIVERGE)

RADIUS OF CONVERGENCE $R = 0$. SERIES CONVERGES AT $x = 0$ ONLY ex. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ FIND RADIUS OF CONV. & VALUES X S.T. SERIES CONV.

RATIO TEST: $$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = |x| \underbrace{\lim_{n\to\infty} \frac{1}{n+1}}_{0} = 0$$

$\rho = 0 < 1$ FOR ALL REAL #'S X.

∴ POWER SERIES CONVERGES ON $(-\infty, \infty)$

RADIUS OF CONVERGENCE $R = \infty$.

THEOREM (TERM-BY-TERM DIFFERENTIATION & INTEGRATION OF POWER SERIES)

SUPPOSE $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$ HAS A RADIUS OF CONV. $R > 0$.

THEN f IS DIFFERENTIABLE AND

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \quad \text{FOR } x \in (a-R, a+R)$$

TERM-BY-TERM DIFFERENTIATION

AND $$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C$$

FOR $x \in (a-R, a+R)$

TERM-BY-TERM INTEGRATION.

ex. $$f(x) = \sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$$ (RADIUS OF CONV. $R=5$ CONVERGES ON $(-8, 2)$.)

NOW WE KNOW f IS DIFFERENTIABLE. ($\Rightarrow$ CONTINUOUS $\Rightarrow$ INTEGRABLE)

$$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{d}{dx}\frac{n(x+3)^n}{5^n}\right]$$

TERM-BY-TERM DIFFERENTIATION.

$$= \sum_{n=1}^{\infty} \frac{n^2(x+3)^{n-1}}{5^n}$$

AND $$\int f(x)\,dx = \int\left(\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}\right)dx$$

$$= \sum_{n=0}^{\infty}\left(\int \frac{n(x+3)^n}{5^n}\,dx\right) = \sum_{n=0}^{\infty} \frac{n(x+3)^{n+1}}{(n+1)5^n}$$

ex. LET $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \boxed{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots}$

(CONVERGES FOR $|x| \leq 1$)

THEN $\underline{f'(x)} = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$= \sum_{n=0}^{\infty} \left(\frac{d}{dx} \frac{(-1)^n x^{2n+1}}{2n+1} \right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{2n+1} \underbrace{\frac{d}{dx}\left[x^{2n+1} \right]} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\cancel{(2n+1)}} \cdot \cancel{(2n+1)} (x^{2n}) = \sum_{n=0}^{\infty} (-x^2)^n$$

$((-1)x^2)^n$

GEOMETRIC SERIES $\quad a = 1 \quad r = -x^2$

CONVERGES TO $\frac{a}{1-r} = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$

$\therefore f'(x) = \frac{1}{1+x^2}$. $\quad \frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$

SO FOR $\underline{-1 < x < 1}$, BOTH $f(x)$ & $\tan^{-1} x$

HAVE THE SAME DERIVATIVE!

TERM-BY-TERM DIFFERENTIATION AND INTEGRATION:

SUPPOSE $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ HAS RADIUS OF CONV. R.

$(a-R, a+R)$

THEN f IS DIFFERENTIABLE AND

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \quad \text{FOR } x \in (a-R, a+R).$$

AND

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} + C$$

FOR $x \in (a-R, a+R)$

ex. 7 LET $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \boxed{x^{2n+1}}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

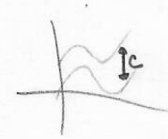
WHICH CONVERGES FOR $-1 \le x \le 1$ $(R = 1)$.

THEN $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{(2n+1)} x^{2n}}{\cancel{2n+1}} = 1 - x^2 + x^4 - x^6 + \dots$

GEOMETRIC SERIES

$a = 1$, $r = -x^2$ $\quad \frac{1}{1-r}$

$$= \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$



$\therefore$ $f(x)$ & $\tan^{-1} x$ HAVE SAME DERIV! $\Rightarrow$ $f(x) = \tan^{-1} x + C$

AND $f(0) = 0 \Rightarrow C = 0 \therefore f(x) = \tan^{-1} x$.

(IF $f' = g'$ THEN $f = g + c$)

$\therefore$ $f(x) = \tan^{-1} x + \underline{\underline{c}}$ FOR $-1 < x < 1$

PLUG-IN $x = 0$: $f(0) = \tan^{-1} 0 + c$

$$\downarrow$$

$$0 = 0 + c \Rightarrow c = 0$$

$\therefore$ $f(x) = \tan^{-1} x$ FOR $-1 < x < 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

CONVERGES TO $\underline{\underline{\tan^{-1} x}}$ FOR $\underline{-1 < x < 1}$

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