§10.8 TAYLOR & MACLAURIN SERIES

RECALL FROM §10.7

- $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ (GEOMETRIC)

  CONVERGES TO $\dfrac{1}{1-x}$ IF $|x| < 1$

  $\therefore$ $\dfrac{1}{1-x}$ (HAS A POWER SERIES REPRESENTATION) $= \displaystyle\sum_{n=0}^{\infty} x^n$

  ON THE INTERVAL $(-1, 1)$

- $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

  CONVERGES TO $\tan^{-1} x$ IF $|x| \leq 1$

  $\therefore$ $\boxed{\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}$ ON $[-1, 1]$

  POWER SERIES REPRESENTATION OF $\tan^{-1} x$ ON $[-1, 1]$

SO THESE FUNCTIONS HAVE POWER SERIES REPRESENTATIONS ON THESE INTERVALS.

RECALL: TERM-BY-TERM DIFFERENTIATION

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n x^n = \underbrace{\sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n x^n\right]}_{\text{ANOTHER POWER SERIES THAT IS DIFFERENTIABLE.}} \quad \text{(POWER RULE)}$$

$\therefore$ ON ITS INTERVAL OF CONVERGENCE, POWER SERIES HAVE DERIVATIVES OF ALL ORDERS: f', f'', f''', etc.

QUESTION: IF A FUNCTION HAS DERIVATIVES OF ALL ORDERS ON AN INTERVAL, DOES IT HAVE A POWER SERIES REPRESENTATION ON AT LEAST PART OF THAT INTERVAL?

(REQUIREMENT: MUST HAVE DERIVATIVES OF ALL ORDERS)

FOR NOW: ASSUME $f(x)$ HAS DERIVATIVES OF ALL ORDERS ON AN INTERVAL CONTAINING $x = a$.

ASSUME $f(x)$ HAS A POWER SERIES REPRESENTATION ON AN INTERVAL CONTAINING $x = a$, WITH A POSITIVE RADIUS OF CONVERGENCE.

RADIUS OF CONVERGENCE

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{ON INTERVAL} \quad (a-R, a+R)$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

WHAT ARE THESE COEFFICIENTS?

$$f(a) = c_0 + \underbrace{c_1(a-a) + c_2(a-a)^2 + c_3(a-a)^3 + \dots}_{0}$$

$$f(a) = c_0$$

$$f'(x) = c_1 + \underbrace{2c_2(x-a) + 3c_3(x-a)^2 + \dots}_{0 \text{ WHEN } x=a}$$

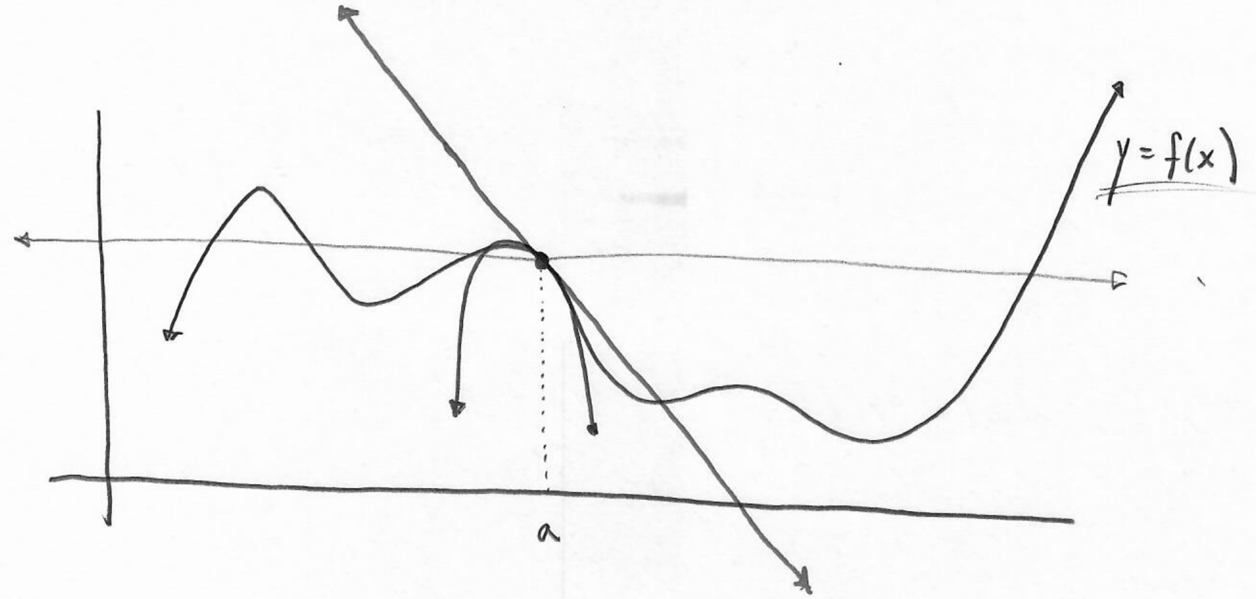
$$f'(a) = c_1$$

$$f''(x) = 2c_2 + \underbrace{3 \cdot 2 c_3 (x-a) + \dots}_{0 \text{ WHEN } x=a}$$

$$f''(a) = 2c_2$$

$$f^{(n)}(x) = n!\, c_n \cancel{(x-a)^0} + (x-a)[\ \dots\dots\ ]$$

$$f^{(n)}(a) = n!\, c_n \Rightarrow \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$



0-DEGREE: $f(x) \approx f(a)$

1-DEGREE: $f(x) \approx f(a) + c_1 (x-a)$

$c_1 = f'(a)$

LINEAR APPROX.

2-DEGREE

$$f(x) \approx f(a) + f'(a)(x-a) + c_2 (x-a)^2$$

$$f'(x) \approx f'(a) + 2c_2 (x-a)$$

$$f''(x) \approx 2c_2 \Rightarrow c_2 = \frac{f''(a)}{2}$$

MAKE IT MATCH AT $x=a$

3-DEGREE

$$f(x) \approx c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3$$

$$f'(x) \approx c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2$$

$$f''(x) \approx 2c_2 + 3 \cdot 2\, c_3 (x-a)$$

$$f'''(x) \approx 3 \cdot 2\, c_3 = 3!\, c_3$$

MATCH AT $x=a$

$$f'''(a) \approx 3!\, c_3 \Rightarrow c_3 = \frac{f'''(a)}{3!}$$

$\therefore$ IF f HAS A POWER SERIES REPRESENTATION AT $x = a$

IT MUST BE

$$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{c_n} (x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$+ \frac{f'''(a)}{3!}(x-a)^3 + \frac{f''''(a)}{4!}(x-a)^4 + \ldots$$

Def: THIS IS CALLED THE TAYLOR SERIES GENERATED BY f AT $x = a$. IF $a = 0$, IT IS ALSO CALLED THE MACLAURIN SERIES FOR f.

Def: $$f(x) \approx \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

IS THE THE TAYLOR POLYNOMIAL OF ORDER N GENERATED BY f AT $x = a$.

(BEST POLYNOMIAL APPROX. TO f WITH DEGREE $\leq N$ IN A NEIGHBORHOOD OF a. $(a-R, a+R)$)

ex. FIND THE TAYLOR SERIES FOR $f(x) = \sin(x)$ AT $x = 0$ (MACLAURIN SERIES).

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad \left(\text{MACLAURIN SERIES } (a=0) \right)$$

n	$f^{(n)}(x)$	$\frac{f^{(n)}(0)}{n!}$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	$-\frac{1}{3!}$
4	$\sin x$	0
5	$\cos x$	$\frac{1}{5!}$
$\vdots$	$\vdots$	$\vdots$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \cancel{0} + \frac{1}{1!} x + \cancel{\frac{0}{2!}} x^2 - \frac{1}{3!} x^3 + \cancel{\frac{0}{4!}} x^4 + \frac{1}{5!} x^5 + \dots$$

$$= \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots}$$

LIST THE FIRST FEW TERMS

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

(NOTICE PATTERN & GENERALIZE)

FIND THE TAYLOR POLYNOMIAL OF ORDER 6 FOR $f(x) = \sin x$.

BEST POLYNOMIAL APPROX TO $f(x) = \sin x$

OF DEGREE ≤ 6.

$$\sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!} x^n = \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!}}$$

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ex. FIND MACLAURIN SERIES FOR $f(x) = 2^x$.

LIST FIRST 4 NONZERO TERMS.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

n	$f^{(n)}(x)$
0	2^x
1	$2^x(\ln 2)$
2	$2^x(\ln 2)^2$
3	$2^x(\ln 2)^3$
⋮	⋮

$$2^x = \frac{1}{0!}x^0 + \frac{\ln 2}{1!}x^1 + \frac{(\ln 2)^2}{2!}x^2 + \frac{(\ln 2)^3}{3!}x^3 + \ldots$$

$$2^x = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n$$

ex. FIND FIRST 3 NON-ZERO TERMS OF MACLAURIN SERIES FOR

$f(x) = x \sin^2(x)$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n} = \frac{6}{3!} x^3 + $$

EVALUATE FOR $n=0$, $n=1$, $n=2$, ...

UNTIL WE GET <u>3 NON-ZERO COEFFICIENTS</u>

$n=0$: $f^{(0)}(x) = f(x) = x \sin^2 x$

$f^{(0)}(0) = 0$

$n=1$: $f^{(1)}(x) = \frac{d}{dx}\left[x \sin^2 x\right]$

$= \sin^2 x + x \cdot 2 \sin x \cos x$

$f^{(1)}(0) = 0$

$n=2$: $f^{(2)}(x) = \frac{d}{dx}\left[\sin^2 x + 2x \sin x \cos x\right]$

$= 2 \sin x \cos x + 2 \sin x \cos x + 2x \cos^2 x$

$- 2x \sin^2 x$

$f^{(2)}(0) = 0$

$n=3:$ $f^{(3)}(x) = \frac{d}{dx}\left[4 \sin x \cos x + 2x(\cos^2 x - \sin^2 x)\right]$

$= \frac{d}{dx}\left[2 \sin 2x + 2x \cos 2x\right]$

$= \underline{4 \cos 2x + 2\cos 2x} - 4x \sin 2x$

$= 6 \cos 2x - 4x \sin 2x$

$f^{(3)}(0) = 6$

$n=4$ $f^{(4)}(x) = \frac{d}{dx}\left[6 \cos 2x - 4x \sin 2x\right]$

$= -12 \sin 2x - 4 \sin 2x - 8x \cos 2x$

$= -16 \sin 2x - 8x \cos 2x$

§10.9 CONVERGENCE OF TAYLOR SERIES

RECALL: IF f HAS DERIVATIVES OF ALL ORDERS ON AN OPEN INTERVAL I CONTAINING $x=a$, THEN f HAS A TAYLOR SERIES ABOUT $x=a$.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

AND TAYLOR POLYNOMIALS OF ORDER n AT $x=a$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

FOR $n = 0, 1, 2, \ldots$ ← APPROXIMATIONS FOR f

QUESTIONS:

1. FOR WHAT VALUES OF x DOES THE TAYLOR SERIES GENERATED BY f ABOUT $x=a$ CONVERGE TO $f(x)$?

2. HOW ACCURATELY DO TAYLOR POLYNOMIALS GENERATED BY f APPROXIMATE f?

TAYLORS THEOREM ON CONVERGENCE OF TAYLOR SERIES / TAYLOR'S FORMULA

IF f HAS DERIVATIVES OF ALL ORDERS IN AN OPEN INTERVAL I CONTAINING a, THEN FOR EACH POSITIVE INTEGER n & FOR EACH $x \in I$

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{\text{TAYLOR POLYNOMIAL OF ORDER } n} + R_n(x)$$

MISSING PIECE TO MAKE EQUALITY.

"REMAINDER" ERROR TERM

WHERE $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

FOR SOME c BETWEEN a & x. $\left(\begin{array}{l} a \leq c \leq x \\ x \leq c \leq a \end{array}\right)$

SPECIAL CASE OF TAYLOR'S THM: MEAN VALUE THM.

$\boxed{f(b) - f(a) = f'(c)(b-a)}$ FOR SOME c BETWEEN a & b.

$f(x) - f(a) = f'(c)(x-a)$

TAYLOR'S FORMULA FOR $n = 1$ $\longrightarrow$ $f(x) = f(a) + f'(c)(x-a)$ FOR SOME c BETWEEN a & x

So $f(x) = P_n(x) + R_n(x)$

TAYLOR POLYNOMIAL — ERROR TERM

THM: IF ERROR $R_n(x) \to 0$ AS $n \to \infty$ FOR ALL $x \in I$ THEN THE TAYLOR SERIES GENERATED BY f AT $x=a$ CONVERGES TO f ON I.

ex. SHOW THAT THE TAYLOR SERIES GENERATED BY $f(x) = \sin x$ AT $x = 0$ (MACLAURIN SERIES) CONVERGES TO $f(x)$ FOR ALL REAL NUMBERS.

WE KNOW $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$$

WHERE $R_n(x) = \frac{f^{(n+1)}(0)}{(n+1)!} x^{n+1}$

$-1 \le f^{(n+1)}(0) \le 1$

$\pm \sin 0$
$\pm \cos 0$

So $0 \leq \left| R_n(x) \right| \leq \frac{1}{(n+1)!} x^{n+1}$

$\therefore 0 \leq \lim_{n\to\infty} \left| R_n(x) \right| \leq \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$ FOR ALL x

$\therefore$ TAYLOR SERIES CONVERGES TO $f(x) = \sin(x)$

FOR ALL x.