§ 10.9 CONVERGENCE OF TAYLOR SERIES

GIVEN $f(x)$ WITH DERIVATIVES OF ALL ORDERS

- INFINITELY DIFFERENTIABLE (SMOOTH)

TAYLOR SERIES FOR $f(x)$ AT $x=a$ IS

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

TAYLOR POLYNOMIAL OF ORDER N FOR f AT $x=a$

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

TAYLORS' FORMULA:

$$f(x) = P_N(x) + R_N(x)$$

ERROR/REMAINDER

MISSING PIECE

FACT:

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

WHERE c IS BETWEEN a & x.

THM: IF $R_N(x) \to 0$ AS $N \to \infty$ FOR ALL $x \in I$

THEN THE TAYLOR SERIES GENERATED BY f AT $x=a$

CONVERGES TO f AT ALL $x \in I$.

ex. $f(x) = e^x$.

1. FIND TAYLOR SERIES FOR f AT $\underbrace{x = 0}_{\text{MACLAURIN SERIES}}$. ✓

2) FIND ALL VALUES x S.T. TAYLOR SERIES CONVERGES. ✓

3) SHOW THAT TAYLOR SERIES CONVERGES TO f ON ITS INTERVAL OF CONVERGENCE. ✓

n^{TH} DERIVATIVE OF f

1. $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Note: $f(x) = f'(x) = f''(x) = \ldots = e^x$

$$1 = \underset{n=0}{f(0)} = \underset{n=1}{f'(0)} = \underset{n=2}{f''(0)} = \ldots$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots}_{\text{TAYLOR SERIES FOR } e^x}$$

2. RATIO TEST: $$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = |x| \underbrace{\lim_{n\to\infty} \left| \frac{1}{n+1} \right|}_{0} = 0$$

$\rho = 0 < 1$ FOR ALL x

$\Rightarrow$ SERIES CONVERGES FOR ALL x.

$R = \infty$, INTERVAL OF CONVERGENCE $(-\infty, \infty)$

ex. FIND $\sin\left(\frac{1}{2}\right)$.

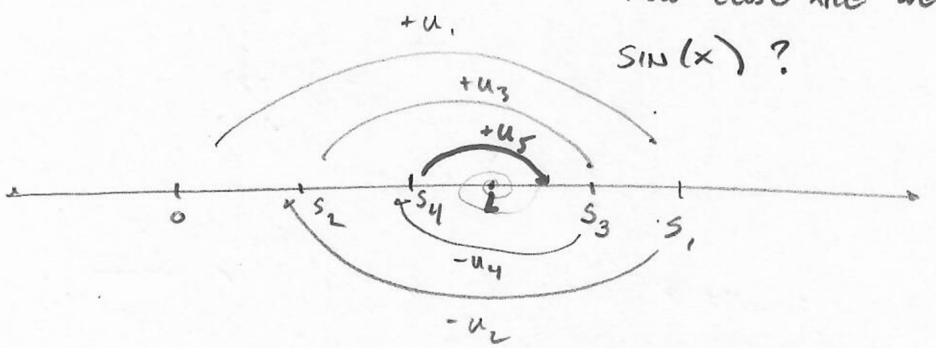
CONSIDER $f(x) = \sin x$

FIND MACLAURIN SERIES : $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$= \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}} + \frac{x^9}{9!} - \cdots$$

IF WE STOP HERE, HOW CLOSE ARE WE TO $\sin(x)$?

NOTE: SERIES IS ALTERNATING.



IF WE STOP AT SUM OF FIRST FOUR TERMS,

OUR ERROR IS NO GREATER THAN THE MAGNITUDE OF THE NEXT TERM : $\frac{x^9}{9}$

$$\sin\left(\frac{1}{2}\right) \approx \left(\frac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^5}{5!} - \frac{\left(\frac{1}{2}\right)^7}{7!} = .\cancel{4634672619} \quad .4794255332$$

$$\text{WITH ERROR} \leq \frac{\left(\frac{1}{2}\right)^9}{9!} = .\cancel{000217013888} \quad .00000137786$$

$$\sin\left(\frac{1}{2}\right) = .4794255386$$

TAYLOR SERIES TO KNOW :

$f(x)$	$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$

MEMORIZE

ex. USE A KNOWN TAYLOR SERIES TO FIND TAYLOR SERIES FOR $\sin(x^2)$.

IDENTITY !

$$\text{TAYLOR SERIES FOR } \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

§ 10.10 APPLICATIONS OF TAYLOR SERIES

.) $\sin\left(\frac{1}{2}\right)$ BY HAND ✓

.) EVALUATE $\int \sin(x^2)\,dx$

$\big\downarrow$ TAYLOR SERIES ABOUT $x=0$

$$\int x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots \, dx$$

TERM BY TERM INTEGRATION !

$$= \frac{1}{3}x^3 - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \dots \quad \left(= \Sigma \right)$$

.) EVALUATE: $\int_0^1 \sin(x^2)\,dx$ WITH ERROR $\leq$ .001

$$= \frac{1}{3}x^3 - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \dots \Big|_0^1$$

.309523...

$$= \underbrace{\frac{1}{3} - \frac{1}{7\cdot 3!}}_{\text{ERROR } .02} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \dots$$

ALTERNATING

ERROR .0008

§10.10 APPLICATIONS OF TAYLOR SERIES

$\int$ ONLY

EVALUATING NON-ELEMENTARY INTEGRALS (NUMERICAL INT FOR DEFINITE INT)

$$\int f(x)\,dx = ?$$

ex. USE TAYLOR SERIES TO EVALUATE

$$\int \sin(x^2)\,dx$$

RECALL: $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

(CONVERGES FOR ALL x)

$\sin(x^2) = (x^2) - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \frac{1}{7!}(x^2)^7 + \dots$

(CONVERGES FOR ALL x)

$$\int \sin(x^2)\,dx = \int x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \dots\,dx$$

$$= C + \underbrace{\frac{1}{3}x^3 - \frac{1}{3!\cdot 7}x^7 + \frac{1}{5!\cdot 11}x^{11} - \frac{1}{7!\cdot 15}x^{15} + \dots}_{\text{CONVERGES FOR ALL } x.}$$

$$\sin^2 x = (\sin x)^2 = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^2$$

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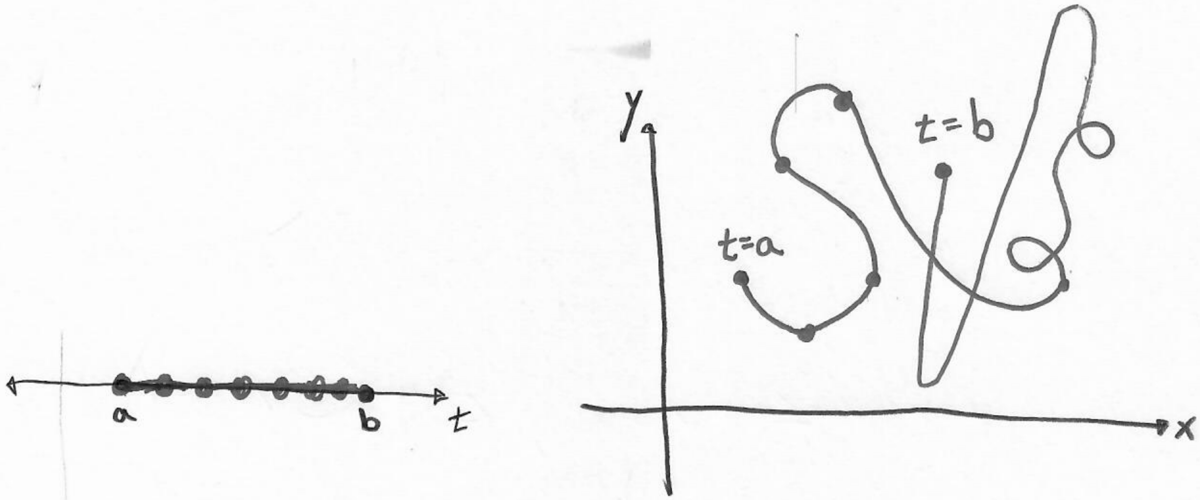
ex. USE KNOWN TAYLOR SERIES TO FIND TAYLOR SERIES FOR

$$f(x) = x^3 e^x = x^3\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)$$

$$= x^3 + x^4 + \frac{x^5}{2!} + \frac{x^6}{3!} + \frac{x^7}{7!} + \dots$$

END OF CH 10

§ 11.1 PARAMETRIZATIONS OF PLANE CURVES



TWO FUNCTIONS OF t

$$x = f(t)$$
$$y = g(t)$$

CONTINUOUS FUNCTIONS DEFINED ON $[a, b]$

$t = a \to (f(a), g(a))$ POINT IN PLANE

$f(a) = 1, \quad g(a) = 2$

$t = b \to (f(b), g(b))$ POINT

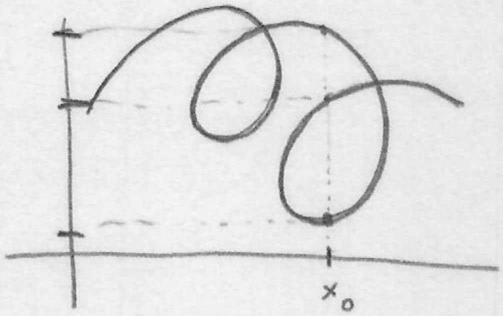
$f(b) = 3 \quad g(b) = 4 \quad (3, 4)$

PARAMETER

$t \longmapsto (f(t), g(t))$ IMAGE IS CALLED PARAMETRIC CURVE.

f CONTROLS LEFT/RIGHT MOTION

g CONTROLS UP/DOWN MOTION.



$y = f(x_0) = ?$

INTRODUCE A THIRD VARIABLE t.

LET BOTH x & y DEPEND ON t

(PARAMETER)

Def: IF x & y ARE GIVEN AS FUNCTIONS

$$(*) \quad x = f(t), \quad y = g(t)$$

OVER AN INTERVAL I OF t-VALUES,

THEN THE SET OF POINTS $(x, y) = (f(t), g(t))$

DEFINED BY THESE EQUATIONS IS A

<u>PARAMETRIC CURVE</u>. THE EQUATIONS $(*)$

ARE CALLED <u>PARAMETRIC EQUATIONS</u> FOR THE CURVE.

t IS CALLED THE <u>PARAMETER</u>.

TO EVERY VALUE FOR t THERE IS A CORRESPONDING POINT ON THE GRAPH

THINK OF PARAMETER t AS TIME.

AND $f(t)$ GIVES THE X-COORD. OF A MOVING PARTICLE AT TIME t, $a \le t \le b$

$g(t)$ GIVES THE y-COORD. OF A MOVING PARTICLE AT TIME t, $a \le t \le b$.

WE SAY $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$; $\underbrace{t \in I}$

t BELONGS TO AN INTERVAL I.

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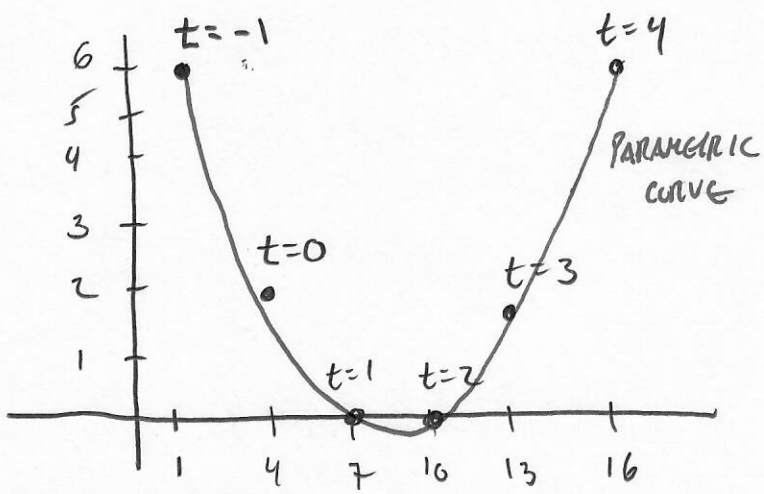
ex. PARAMETRIC CURVE: $x = 4 + 3t$

$y = t^2 - 3t + 2$; $-1 \le t \le 4$

SKETCH THE PARAMETRIC CURVE & THEN "ELIMINATE THE PARAMETER"

PLOT:

t	x	y
-1	1	6
0	4	2
1	7	0
2	10	0
3	13	2
4	16	6



TO SHOW THE CURVE <u>IS</u> A PARABOLA, WE "ELIMINATE THE PARAMETER"

$$x = 4 + 3t \xrightarrow{\text{SOLVE FOR } t} t = \frac{x-4}{3}$$

$$y = t^2 - 3t + 2$$

PLUG IN (SUBSTITUTE)

$$\underbrace{y = \left(\frac{x-4}{3}\right)^2 - 3\left(\frac{x-4}{3}\right) + 2}$$

QUADRATIC EQ

GRAPH: PARABOLA ✓

HW FINISH UP THROUGH §10.10

QUIZ TUE. 10.7, 10.8, 10.9, 10.10

(§11.1 IF YOU WANT TO GET A HEAD START)