# §2.4 The Derivative as a Function

RECALL: **Derivative of $f$ at $a$** (fixed number)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

1. Decide where to find derivative, $a$
2. Evaluate limit $f'(a)$
↳ NUMBER

**Def:** **Derivative of $f$**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

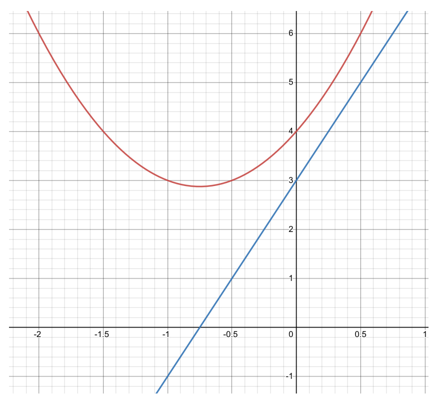
is the function that gives the derivative of $f$ at any point $x$, where $f$ is differentiable.

1. Evaluate limit $f'(x)$
2. Plug in value for $x$
↳ FUNCTION

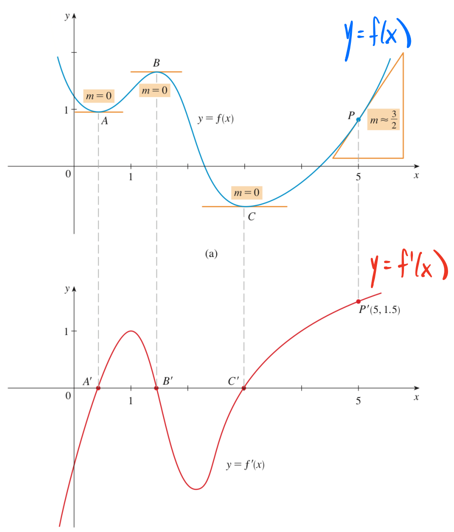
**ex.** Let $f(x) = 2x^2 + 3x + 4$.

(a) Find $f'(x)$.
(b) Find $f'(-1)$ & $f'(2)$.
(c) For what value(s) of $x$ is $f'(x) = 0$?

https://www.desmos.com/calculator/rtz7qbfs64




**Note:** If $f'(x) > 0$ on an interval then $f(x)$ is increasing on that interval.
If $f'(x) < 0$ on an interval then $f(x)$ is decreasing on that interval.




**ex.** (a) Find where $f'(x) = 0$.
(b) On which intervals is $f'(x)$ pos/neg?
(c) Sketch $f'(x)$.

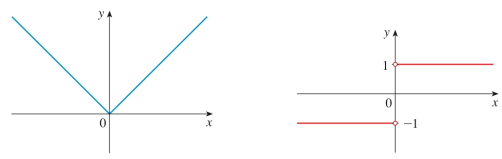
The slope of this graph is the height of this graph

If $f'(x)$ changes sign NEG → POS at $x = a$ then $f(a)$ is a local MIN value of $f$.
If $f'(x)$ changes sign POS → NEG at $x = a$ then $f(a)$ is a local MAX value of $f$.

**ex.** (a) SKETCH THE GRAPH OF $f(x) = |x|$ AND $f'(x)$.

(b) COMPARE DOMAIN OF $f(x)$ & $f'(x)$.

( NOTE THAT $f(x) = |x|$ HAS LOCAL MIN VALUE 0 AT $x = 0$. )




(a) $y = f(x) = |x|$ (b) $y = f'(x)$

**NOTE:** WHEN $f$ IS DIFFERENTIABLE AT $a$, WHEN WE ZOOM IN ON GRAPH $y = f(x)$ AT $(a, f(a))$, IT LOOKS LIKE A LINE WITH DEFINED SLOPE ("SMOOTH") THAT SLOPE IS $f'(a)$.

https://www.desmos.com/calculator/hlbpzlmjyy

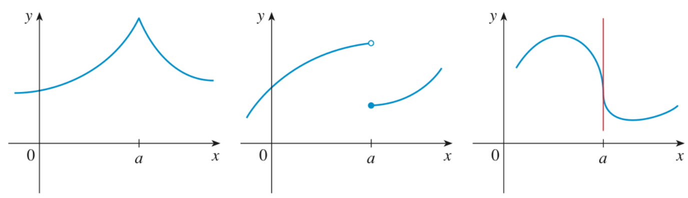



**FIGURE 6** Three ways a derivative can fail to exist

(a) A corner (b) A discontinuity (c) A vertical tangent

**NOTE:** $f(x) = |x|$ IS CONTINUOUS AT $x = 0$ BUT NOT DIFFERENTIABLE AT $x = 0$.

DRAW WITHOUT LIFTING PENCIL NOT SMOOTH

THE OPPOSITE IS IMPOSSIBLE:

**(3) Theorem** If a function is differentiable at a number, then it is continuous there.

**PROOF:**

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left[ f(a) + \frac{f(x) - f(a)}{x - a} (x - a) \right]$$

$$= \lim_{x \to a} \left[ f(a) \right] + \lim_{x \to a} \left[ \frac{f(x) - f(a)}{x - a} \right] \cdot \left[ \lim_{x \to a} (x - a) \right]$$

$$= f(a) + f'(a) \cdot 0 = f(a) \quad \checkmark$$

## ALTERNATIVE NOTATION

GIVEN $f(x)$, WE DENOTE ITS DERIVATIVE AS

$f'(x)$, $\frac{d}{dx} f(x)$, OR JUST $\frac{df}{dx}$.

WHEN WE SET $y = f(x)$, WE ALSO HAVE

$y'$ OR $\frac{dy}{dx}$

ALL MEAN THE SAME THING.

RECALL: AVERAGE RATE OF CHANGE $= \frac{\Delta y}{\Delta x}$

& INSTANTANEOUS RATE OF CHANGE

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

WHEN USING THIS NOTATION,

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

"EVALUATED AT"

ex. SUPPOSE $y = \frac{1}{\sqrt{x}}$.

(a) FIND $\frac{dy}{dx}$ & $\left.\frac{dy}{dx}\right|_{x=4}$.

(b) SHOW $f$ IS DECREASING ON $(0,\infty)$

(a)

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)} = \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x+0}\left(\sqrt{x} + \sqrt{x+0}\right)} = \frac{-1}{2x\sqrt{x}}$$

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{-1}{2\sqrt{4}(4)} = \frac{-1}{16}$$

(b) SINCE $\frac{-1}{2x\sqrt{x}} < 0$ ON INTERVAL $(0,\infty)$,

$\frac{1}{\sqrt{x}}$ IS DECREASING ON INTERVAL $(0,\infty)$.

## HIGHER DERIVATIVES

GIVEN $f(x)$ WITH DERIVATIVE $f'(x)$,
THE SECOND DERIVATIVE OF $f(x)$ IS THE DERIVATIVE OF $f'(x)$, DENOTED $f''(x)$.

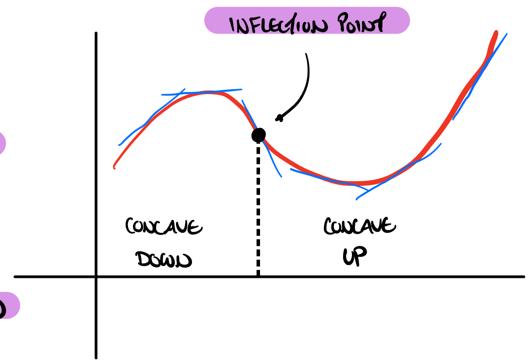
( OR, IF $y = f(x)$, WE MAY WRITE $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ )

$f''(x)$ GIVES THE INSTANTANEOUS RATE OF CHANGE OF $f'(x)$ WITH RESPECT TO $x$.

e.g. IF $s(t)$ = POSITION OF MOVING OBJECT ON A STRAIGHT TRACK,
THEN $s'(t) = \frac{ds}{dt} = v(t)$ IS THE VELOCITY OF THE OBJECT, AND
$s''(t) = \frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}v(t) = v'(t) = a(t)$ IS THE ACCELERATION OF THE OBJECT.

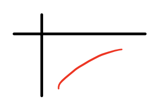
POSITIVE ACCELERATION $\Rightarrow$ VELOCITY IS INCREASING
$f''(x) > 0 \Rightarrow f'(x)$ IS INCREASING
$\Rightarrow f(x)$ IS CONCAVE UP

NEGATIVE ACCELERATION $\Rightarrow$ VELOCITY IS DECREASING
$f''(x) < 0 \Rightarrow f'(x)$ IS DECREASING
$\Rightarrow f(x)$ IS CONCAVE DOWN

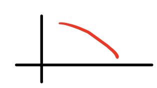



ex. SKETCH A CURVE SUCH THAT

(a) $f(x) < 0$, $f'(x) > 0$, $f''(x) < 0$.



(b) $f(x) > 0$, $f'(x) < 0$, $f''(x) < 0$.



(c) $f(x) < 0$ $f'(x) < 0$, $f''(x) > 0$