§3.6 Exponential Growth & Decay

Any quantity that grows or declines at a rate proportional to itself can be modeled by an exponential function

$$f(x) = ca^x, \quad a > 0$$

$$\downarrow$$

$$f'(x) = ca^x \cdot \ln a$$

Constant Percentage Growth

ex. An investment of \$10,000 grows 4% per year.

(a) Give expression for value of investment after t years.

(What if it shrinks by 4% per year?)

(b) How many years before it is worth \$12,000? 4.6486 yrs

(c) If interest is compounded monthly, how many years before it is worth \$12,000? 4.5656 yrs

(d) If interest is compounded continuously, how many years before it is worth \$12,000? 4.5580 yrs

An investment of P dollars earning annual interest rate r for t years

$$\boxed{A(t) = P(1+r)^t}$$

When interest is compounded n times per year

$$\boxed{A(t) = P\left(1+\frac{r}{n}\right)^{nt}}$$

When interest is compounded continuously

$$\boxed{A(t) = Pe^{rt}}$$

Proof:

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = P\cdot\lim_{n\to\infty}\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$$

$$\left(\text{let } x = \frac{r}{n}\right) \quad = P\cdot\lim_{x\to 0}\left[(1+x)^{1/x}\right]^{rt} = Pe^{rt}$$

ex. Compute the interest earned on \$25,000 investment earning 3.6% interest

(a) compounded quarterly
(b) compounded continuously

Law of Natural Growth:

Rate of growth is proportional to size of population.

$P'(t) = k\,P(t) \longrightarrow \boxed{P(t) = Ce^{kt}}$

$P'(t) = kCe^{kt} = k\,P(t)$ ✓

k is called relative growth rate. $C = P(0)$

e.g. When population is 4,000, 5,000, 6,000, population grows at rate of 400, 500, 600 per year

$P'(t) = .1\,P(t) \sim P(t) = Ce^{.1t}$

$P'(t) = Ce^{.1t}(.1) = .1\,P(t)$ ✓

relative growth rate $k = .1$

ex. A bacteria culture has initial population 100 and grows to 230 after one hour. Find model for population after t hours? What is the relative growth rate?

Radioactive Decay

Radioactive material loses mass (by spontaneously emitting radiation) at a rate proportional to its mass.

Natural growth with negative relative growth rate

$$\boxed{M(t) = Ce^{kt} \quad (k < 0)}$$

Def: THE HALF-LIFE OF A RADIOACTIVE MATERIAL IS THE TIME IT TAKES TO LOSE HALF ITS MASS. OFTEN DENOTED λ "LAMBDA".

ex. FIND REMAINING MASS AFTER t YEARS OF A SAMPLE OF RADIOACTIVE MATERIAL WITH HALF-LIFE λ AND INITIAL MASS m_0 GRAMS.
WHAT IS THE RELATIVE GROWTH RATE?

$$\left.\begin{array}{l} M(t) = Ce^{kt} \\ M(0) = m_0 \end{array}\right\} M(t) = m_0 e^{kt}$$

$$M(\lambda) = \frac{1}{2} m_0$$

$$\left.\right\} \frac{1}{2} m_0 = m_0 (e^k)^\lambda$$

$$\left(\frac{1}{2}\right)^{1/\lambda} = (e^k)$$

$$\therefore \boxed{M(t) = m_0 \left(\frac{1}{2}\right)^{t/\lambda}}$$

RELATIVE GROWTH RATE: $\left(\frac{1}{2}\right)^{1/\lambda} = e^k$

$$\boxed{k = \frac{1}{\lambda} \ln\left(\frac{1}{2}\right)}$$

EXAMPLE 5

A Model for the Amount of a Radioactive Substance and Its Half-Life

The half-life of the radioactive material radium-226 ($^{226}_{88}Ra$) is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of $^{226}_{88}Ra$ that remains after t years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

(d) At what rate is the mass decreasing when 30 mg remains?