

§ 5.1 Cost, Area, & The Definite Integral

Cost

■ EXAMPLE 1 Estimating a Total Cost

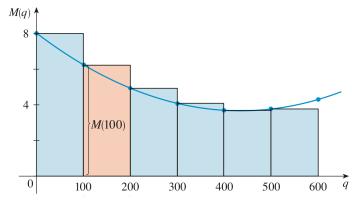
Suppose a soft drink manufacturer estimates that the marginal cost for its cola at different production levels is as given in the following table. Estimate the cost, beyond the fixed costs, to produce the first 600 cases.

Production level (cases)	0	100	200	300	400	500	600
Marginal cost (\$/case)	8.00	6.23	4.92	4.07	3.68	3.75	4.28

MARGINAL COST VARIES.
ESTIMATE TOTAL COST IN BATCHES
OF 100 CASES.

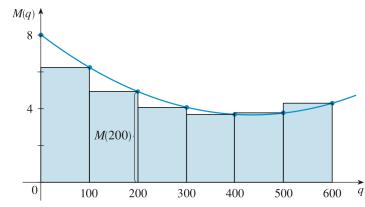
INITIAL VALUE

$$\begin{aligned} \text{BATCH 1: } & (8.00 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$800 \\ \text{BATCH 2: } & (6.23 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$623 \\ \text{BATCH 3: } & (4.92 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$492 \\ \text{BATCH 4: } & (4.07 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$407 \\ \text{BATCH 5: } & (3.68 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$368 \\ \text{BATCH 6: } & (3.75 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$375 + \\ & \qquad \qquad \qquad \$3,065 \end{aligned}$$



TERMINAL VALUE

$$\begin{aligned} \text{BATCH 1: } & (6.23 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$623 \\ \text{BATCH 2: } & (4.92 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$492 \\ \text{BATCH 3: } & (4.07 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$407 \\ \text{BATCH 4: } & (3.68 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$368 \\ \text{BATCH 5: } & (3.75 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$375 \\ \text{BATCH 6: } & (4.18 \text{ DOLLARS/CASE}) \times (100 \text{ CASES}) = \$418 + \\ & \qquad \qquad \qquad \$2,693 \end{aligned}$$



BOTH SUMS ARE ESTIMATES OF THE TOTAL COST OF 600 UNITS.

ESTIMATES WOULD GET BETTER WITH MORE, SMALLER BATCHES.

GRAPHICALLY, THIS WOULD PRODUCE MORE, NARROWER RECTANGLES,
WHICH WOULD BETTER ESTIMATE THE AREA UNDER THE MARGINAL COST CURVE.

THE EXACT COST OF THE 600 UNITS IS THE EXACT AREA UNDER
THE MARGINAL COST CURVE.

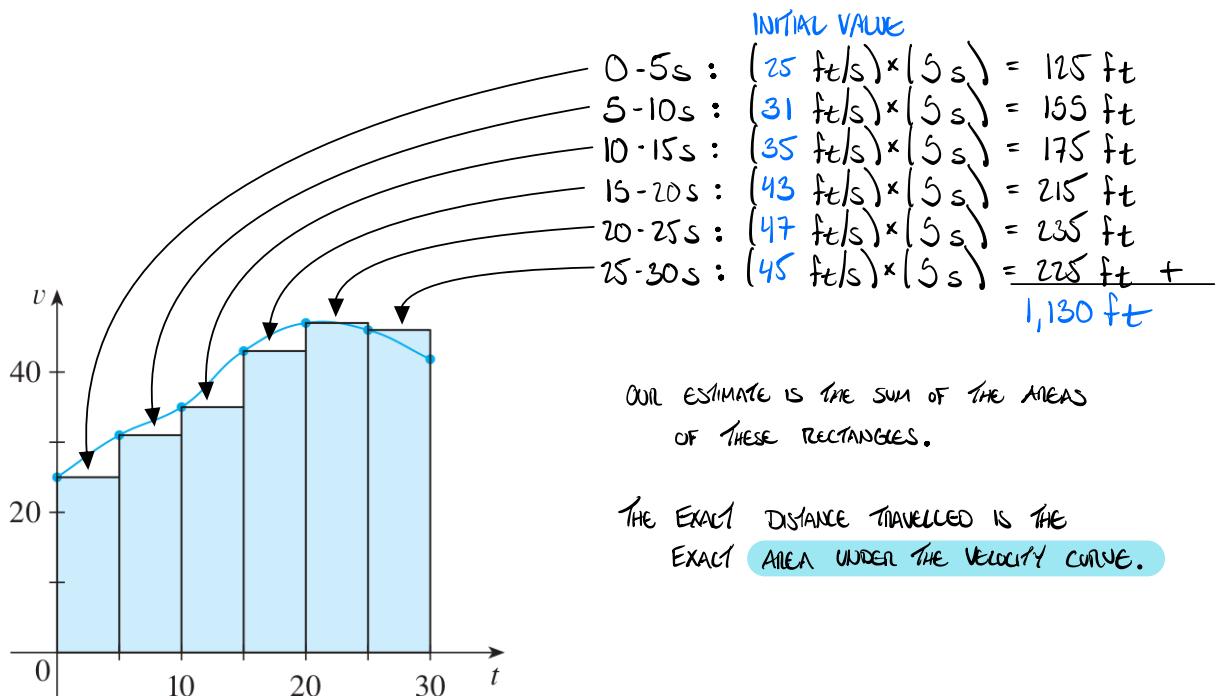
DISTANCE

EXAMPLE 4 Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

Velocity varies. Estimate total distance travelled over 5 second intervals.

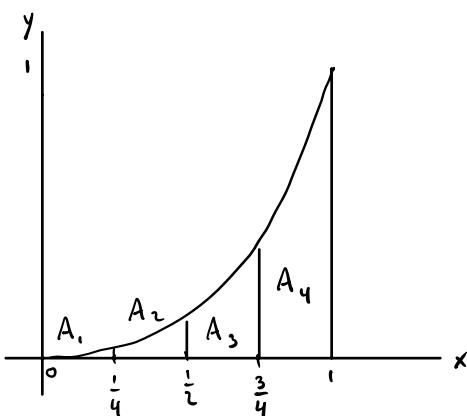
$$\text{Velocity} = \frac{\text{DISTANCE}}{\text{TIME}} \Rightarrow \text{DISTANCE} = \text{VELOCITY} \times \text{TIME}$$



Area

<https://www.geogebra.org/m/nmw6Dhdk>

ex. CALCULATE THE AREA UNDER THE CURVE $y = x^2$ AND ABOVE THE X-AXIS WITH $0 \leq x \leq 1$.



IF WE CUT THE INTERVAL $[0, 1]$ INTO

$$4 \text{ subintervals } I_1 = [0, \frac{1}{4}]$$

$$I_2 = [\frac{1}{4}, \frac{1}{2}]$$

$$I_3 = [\frac{1}{2}, \frac{3}{4}]$$

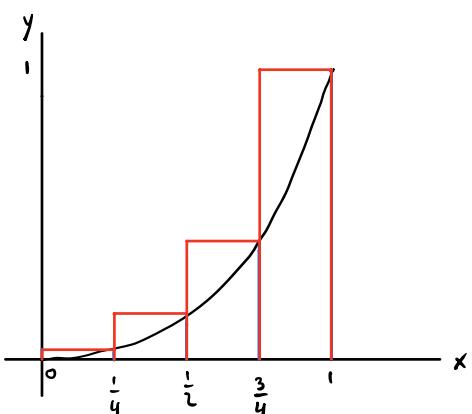
$$I_4 = [\frac{3}{4}, 1]$$

THE TOTAL AREA IS THE SUM OF THE AREAS
UNDER THE CURVE ABOVE EACH SUBINTERVAL.

$$A = A_1 + A_2 + A_3 + A_4$$

WE CAN APPROXIMATE THE AREA A_i OF EACH VERTICAL STRIP AS THE AREA OF A
RECTANGLE WITH BASE $\frac{1}{4}$

AND HEIGHT EQUAL TO THE HEIGHT OF THE CURVE AT THE **RIGHT ENDPOINT**
OF EACH SUBINTERVAL



$$A_1 \approx (\frac{1}{4})^2 (\frac{1}{4}) = \frac{1}{64}$$

$$A_2 \approx (\frac{1}{2})^2 (\frac{1}{4}) = \frac{4}{64}$$

$$A_3 \approx (\frac{3}{4})^2 (\frac{1}{4}) = \frac{9}{64}$$

$$A_4 \approx (1)^2 (\frac{1}{4}) = \frac{16}{64}$$

$$\Rightarrow A = A_1 + A_2 + A_3 + A_4$$

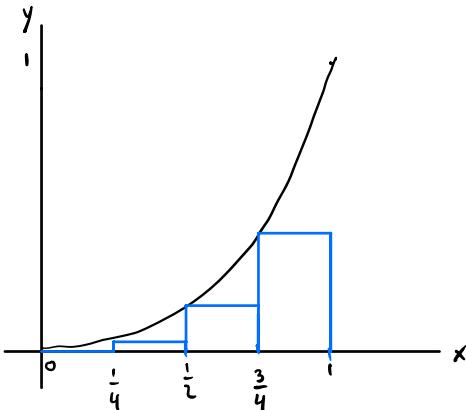
$$A \approx \frac{1}{64} + \frac{4}{64} + \frac{9}{64} + \frac{16}{64} = \frac{30}{64}$$

$$A \approx 0.46875 \quad (\text{OVERESTIMATE})$$

<https://www.geogebra.org/m/nmw6Dhdk>

← CALL THIS R_4

Alternatively, we could have approximated the area of each vertical strip as a rectangle with base $\frac{1}{4}$ and height equal to the height of the curve at the **left** endpoint of each subinterval.



$$\begin{aligned}A_1 &\approx (0)^2 \left(\frac{1}{4}\right) = 0 \\A_2 &\approx \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right) = \frac{1}{64} \\A_3 &\approx \left(\frac{1}{2}\right)^2 \left(\frac{1}{4}\right) = \frac{4}{64} \\A_4 &\approx \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right) = \frac{9}{64}\end{aligned}$$

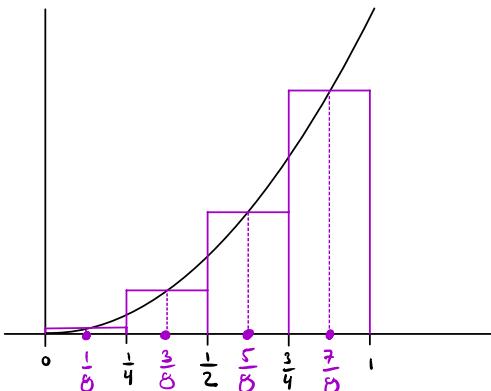
<https://www.geogebra.org/m/nmw6Dhdk>

$$\Rightarrow A = A_1 + A_2 + A_3 + A_4$$

$$A \approx 0 + \frac{1}{64} + \frac{4}{64} + \frac{9}{64} = \frac{14}{64}$$

$A \approx 0.21875$ (underestimate) ← call this L_4

Yet a third way would be to approximate the area of each vertical strip as a rectangle with base $\frac{1}{4}$ and height equal to the height of the curve at the **midpoint** of each subinterval.



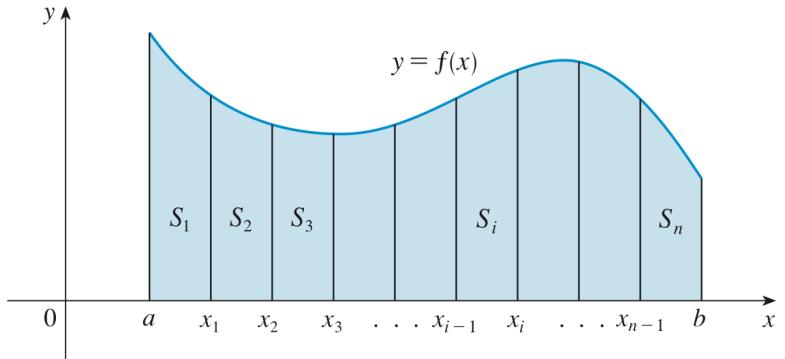
$$\begin{aligned}A_1 &\approx \left(\frac{1}{8}\right)^2 \left(\frac{1}{4}\right) = \frac{1}{256} \\A_2 &\approx \left(\frac{3}{8}\right)^2 \left(\frac{1}{4}\right) = \frac{9}{256} \\A_3 &\approx \left(\frac{5}{8}\right)^2 \left(\frac{1}{4}\right) = \frac{25}{256} \\A_4 &\approx \left(\frac{7}{8}\right)^2 \left(\frac{1}{4}\right) = \frac{49}{256}\end{aligned}$$

$$A = A_1 + A_2 + A_3 + A_4 \\A \approx \frac{1}{256} + \frac{9}{256} + \frac{25}{256} + \frac{49}{256} = \frac{84}{256} = .328125$$

<https://www.geogebra.org/m/nmw6Dhdk>

More Generally

IN GENERAL HERE IS HOW WE CAN CALCULATE THE AREA UNDER THE CURVE $y=f(x)$, ABOVE THE X-AXIS, AND BETWEEN THE VERTICAL LINES $x=a$ & $x=b$.

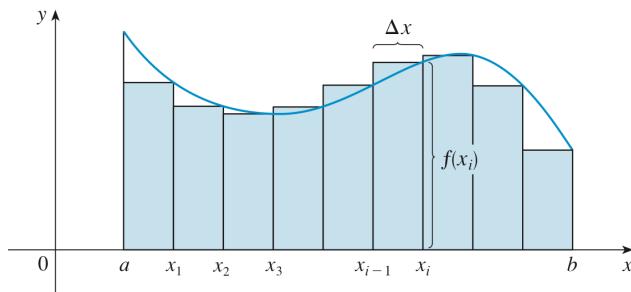


FIRST BREAK $[a,b]$ INTO n SUBINTERVALS,

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n] \quad \begin{matrix} \parallel \\ a \end{matrix} \qquad \begin{matrix} \parallel \\ b \end{matrix}$$

EACH WITH LENGTH $\Delta x = \frac{b-a}{n}$.

THEN APPROXIMATE THE AREA OF THE i^{th} STRIP BY A RECTANGLE WITH WIDTH Δx AND HEIGHT $f(x_i)$, WHICH IS THE VALUE OF f AT THE RIGHT ENDPOINT.



Thus, the area of the i^{th} rectangle is $f(x_i) \Delta x$.

THIS IS CALLED A
RIEMANN SUM

THE SUM OF THE AREAS OF THESE n RECTANGLES IS

$$R_n = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + \dots + f(x_n) \Delta x.$$

(4) ■ Definition The **area** of the region S that lies under the graph of the continuous positive function f is the limit of the sum of the areas of approximating rectangles:

$$\text{area} = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

The Definite Integral

(5) ■ Definition of a Definite Integral If f is a continuous function for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a), x_1, x_2, \dots, x_n (= b)$ be the endpoints of these subintervals. Then the **definite integral of f from a to b** is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

- $f(x)$ is called the **integrand**
- a and b are called the **limits of integration**
- a is the **lower limit**, b is the **upper limit**

■ Definite Integral as a Net Area

$$\int_a^b f(x) dx = A_1 - A_2$$

where A_1 is the area of the region above the x -axis and below the graph of f , and A_2 is the area of the region below the x -axis and above the graph of f .

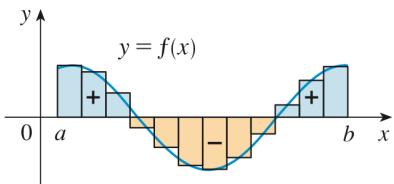


FIGURE 12

$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$ is an approximation to the net area.

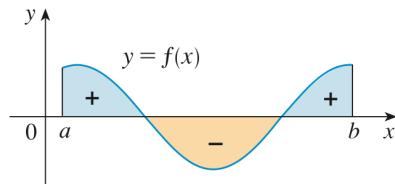


FIGURE 13

$\int_a^b f(x) dx$ is the net area.

■ EXAMPLE 4 Using Geometry to Evaluate Integrals

Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1-x^2} dx$

(b) $\int_0^3 (x-1) dx$

(a) Let $y = \sqrt{1-x^2}$. Note that $y \geq 0$.

Then $y^2 = 1-x^2$, i.e., $x^2+y^2 = 1$, $y \geq 0$

