

§5.2 The Fundamental Theorem of Calculus

Def: A function $F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$

$$F(x) \xrightarrow{\text{ANTI-DERIVATIVE}} f(x) = F'(x) \xrightarrow{\text{DERIVATIVE}}$$

ex. since $\frac{d}{dx} \left[x^3 + \ln x + e^{2x} \right] = 3x^2 + \frac{1}{x} + 2e^{2x}$,

the antiderivative of $3x^2 + \frac{1}{x} + 2e^{2x}$ is $x^3 + \ln x + e^{2x}$.

$$\underbrace{3x^2 + \frac{1}{x} + 2e^{2x}}_{f(x)} \quad \underbrace{x^3 + \ln x + e^{2x}}_{F(x)}$$

$$F'(x) = f(x) \quad \checkmark$$

CONVENTION: CAPITAL LETTERS INDICATE ANTIDERIVATIVES.

■ **The Fundamental Theorem of Calculus** If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is any antiderivative of f , that is, $F' = f$.

ex.

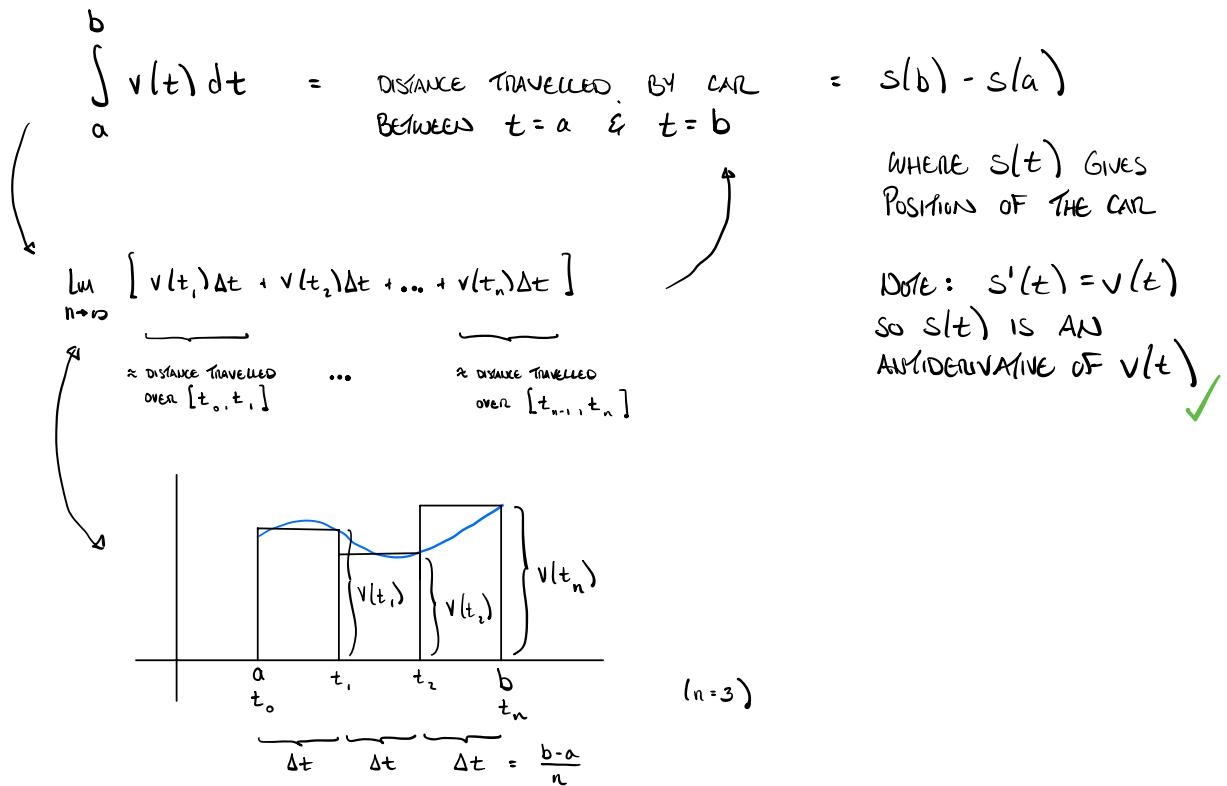
CHECK THAT $F(x) = \sqrt{x} + \ln x$
IS AN ANTIDERIVATIVE OF $f(x) = \frac{1}{2\sqrt{x}} + \frac{1}{x}$.

USE FTC TO EVALUATE THE DEFINITE INTEGRAL $\int_1^4 \left(\frac{1}{2\sqrt{x}} + \frac{1}{x} \right) dx$.

$$\begin{aligned} & \text{FTC} \\ & \int_1^4 f(x) dx = F(4) - F(1) = (2 + \ln 4) - (1 + 0) \\ & = 1 + \ln 4. \end{aligned}$$

Why?

RECALL THAT WHEN $v(t) \geq 0$ IS THE VELOCITY OF A CAR MOVING ALONG A STRAIGHT TRACK



WHERE $s(t)$ GIVES POSITION OF THE CAR

NOTE: $s'(t) = v(t)$

so $s(t)$ IS AN

ANTIDERIVATIVE OF $v(t)$

ANTIDERIVATIVES

Ex.

FIND AN ANTIDERIVATIVE OF

(a) $f(x) = 3x^2$

$F(x) = x^3$

+ C

(b) $f(x) = x^2$

$F(x) = \frac{1}{3}x^3$

+ C

(c) $f(x) = 6x^4$

$F(x) = \frac{6}{5}x^5$

+ C

(d) $f(x) = e^x$

$F(x) = e^x$

+ C

(e) $f(x) = e^{5x}$

$F(x) = \frac{1}{5}e^{5x}$

+ C

WHERE C IS ANY REAL NUMBER (constant)

(1) ■ **Theorem** If F is an antiderivative of f on an interval, then the most general antiderivative of f on that interval is

$$F(x) + C$$

where C is an arbitrary constant.

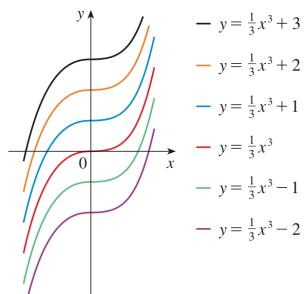


FIGURE 1
Members of the family of
antiderivatives of $f(x) = x^3$

(2) Table of Derivative and Antiderivative Formulas

Function	Derivative	Function	General Antiderivative
$cf(x)$	$cf'(x)$	$cf(x)$	$cF(x) + C$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$
cx	c	c	$cx + C$
x^n ($n \neq 0$)	nx^{n-1}	x^n ($n \neq -1$)	$\frac{x^{n+1}}{n+1} + C$
$\ln x$	$1/x$	$1/x$	$\ln x + C$
e^x	e^x	e^x	$e^x + C$
e^{kx}	ke^{kx}	e^{kx}	$\frac{1}{k}e^{kx} + C$
a^x	$a^x \ln a$	a^x	$\frac{1}{\ln a}a^x + C$

CHECK THESE!

ex. FIND ALL FUNCTIONS $F(x)$ SUCH THAT $F'(x) = 10x + 3\sqrt[3]{x} - \frac{6}{x}$.

$$F(x) = 5x^2 + 2x^{3/2} - 6\ln|x| + C$$

ex. FIND THE FUNCTION $F(x)$ SUCH THAT $F'(x) = 10x + 3\sqrt[3]{x} - \frac{6}{x}$

$$\text{AND } F(1) = 15.$$

$$F(x) = 5x^2 + 2x^{3/2} - 6\ln|x| + 8$$

ex. FIND THE MOST GENERAL ANTIDERIVATIVE OF $f(x) = e^{8x} + 3 + \frac{7}{x^2}$

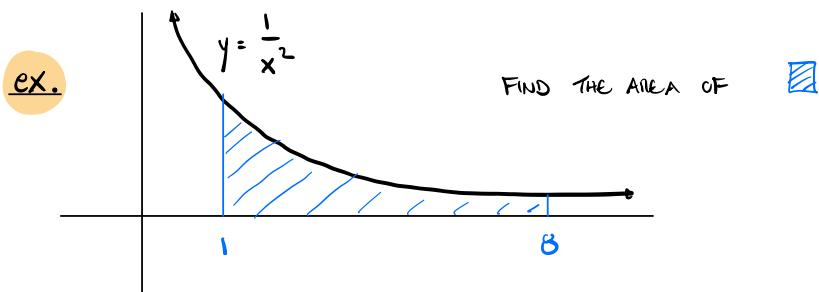
$$F(x) = \frac{1}{8}e^{8x} + 3x - 7x^{-1} + C$$

USING THE FUNDAMENTAL THM OF CALCULUS (FTC)

$$\int_a^b f(x) dx = F(b) - F(a), \text{ where } F'(x) = f(x) \\ (F \text{ IS AN ANTIDERIVATIVE OF } f)$$

NOTATION $F(b) - F(a) = \underbrace{F(x)}_{a}^b = [F(x)]_a^b = [F(x)]_a^b$

ex. EVALUATE $\int_1^2 (6x^2 + 4 - \frac{3}{x}) dx = 2x^3 + 4x - 3\ln x \Big|_1^2$
 $= (16 + 8 - 3\ln 2) - (2 + 4 - 0)$
 $= 18 - 3\ln 2$



$$\text{AREA} = \int_1^8 x^{-2} dx = -x^{-1} \Big|_1^8 = -8^{-1} - (-1^{-1}) = -\frac{1}{8} + 1 = \frac{7}{8}$$

INDEFINITE INTEGRALS

$$\boxed{\int f(x) dx = F(x) \text{ means } F'(x) = f(x)}$$

NO LIMITS OF INTEGRATION

Note: DEFINITE INTEGRAL: $\int_a^b f(x) dx$ IS A NUMBER $F(b) - F(a)$

INDEFINITE INTEGRAL: $\int f(x) dx$ IS A COLLECTION OF FUNCTIONS $F(x) + C$

ex. FIND $\int \frac{(x+3)^2}{x} dx$

$$= \int \frac{x^2 + 6x + 9}{x} dx = \int x + 6 + \frac{9}{x} dx = \frac{1}{2}x^2 + 6x + 9\ln|x| + C$$

ex. FIND $\int (e^{4x} + 8) dx = \frac{1}{4}e^{4x} + 8x + C$

Properties of INDEFINITE INTEGRAL

(1) $\int c f(x) dx = c \int f(x) dx$

(2) $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$

Properties of DEFINITE INTEGRAL

(1) $\int_a^b c f(x) dx = c \int_a^b f(x) dx$

(2) $\int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

(3) $\int_a^b f(x) dx = - \int_b^a f(x) dx$

\downarrow \downarrow

$$\lim_{n \rightarrow \infty} \left[f(x_1) \Delta x + \dots + f(x_n) \Delta x \right], \quad \Delta x = \frac{a-b}{n}$$

$= \lim_{n \rightarrow \infty} \left[f(x_1) \Delta x + \dots + f(x_n) \Delta x \right], \quad \Delta x = \frac{b-a}{n} = - \frac{a-b}{n}$

OPPOSITES!

(4) $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$

