Exam 2

Answer all 10 questions for a total of 100 points. Write your solutions in the accompanying blue book, and put a box around your final answers. Your answers may be left as expressions involving square roots, logarithms, exponentials, etc. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. Find the derivative.

(a) (8 points) $f(x) = \sqrt{x}\ln(x^2+1)$

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[\sqrt{x}]\ln(x^2+1) + \sqrt{x}\frac{d}{dx}[\ln(x^2+1)] \quad \text{(product rule)} \\ &= \frac{1}{2\sqrt{x}}\ln(x^2+1) + \sqrt{x}\frac{1}{x^2+1}\frac{d}{dx}[x^2+1] \quad \text{(chain rule)} \\ &= \frac{1}{2\sqrt{x}}\ln(x^2+1) + \sqrt{x}\frac{2x}{x^2+1} \end{aligned}$$

(b) (8 points) $g(x) = \dfrac{e^{8x}}{(3x^2+5)^2}$

Solution:

$$\begin{aligned} g'(x) &= \frac{\frac{d}{dx}[e^{8x}](3x^2+5)^2 - e^{8x}\frac{d}{dx}[(3x^2+5)^2]}{((3x^2+5)^2)^2} \quad \text{quotient rule} \\ &= \frac{e^{8x}\frac{d}{dx}[8x](3x^2+5)^2 - e^{8x}2(3x^2+5)\frac{d}{dx}[3x^2+5]}{(3x^2+5)^4} \quad \text{chain rule} \\ &= \frac{8e^{8x}(3x^2+5)^2 - 12xe^{8x}(3x^2+5)}{(3x^2+5)^4} = \frac{4e^{8x}(6x^2-3x+10)}{(3x+5)^3} \end{aligned}$$

(c) (8 points) $h(x) = 120\left(\dfrac{1}{3}\right)^{x/4}$

Solution:

$$\begin{aligned} h'(x) &= 120\left(\frac{1}{3}\right)^{x/4}\ln\left(\frac{1}{3}\right)\frac{d}{dx}[x/4] \quad \text{(chain rule)} \\ &= 30\left(\frac{1}{3}\right)^{x/4}\ln\left(\frac{1}{3}\right) \end{aligned}$$

2. (8 points) A caterer estimates that it costs

$$C(q) = .02q^2 + 8q + 375$$

dollars to provide dinner to q guests at a charity event. How large of an event (how many guests) will minimize the average cost per guest?

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Solution: The average cost per guest will be minimized when the number of guests q is such that the average cost is equal to the marginal cost. The average cost and marginal cost are

$$\frac{C(q)}{q} = \frac{.02q^2 + 8q + 375}{q} \quad \text{and} \quad C'(q) = .04q + 8,$$

respectively. Setting these two functions equal and solving for q, we have

$$\begin{aligned} \frac{.02q^2 + 8q + 375}{q} &= .04q + 8 \\ .02q^2 + 8q + 375 &= .04q^2 + 8q \\ 375 &= .02q^2 \\ q &= \sqrt{\frac{375}{.02}} \end{aligned}$$

(FYI, this is ≈ 137 guests.)

3. (4 points) Suppose $C(q)$ is the cost a company must pay to produce q units. If $C(10,500) = 8,200$ and $C'(10,500) = 0.5$, approximately how much would it cost the company to produce 11,000 units?

Solution: We are given that it costs \$8,200 to produce 10,500 units, and each additional unit after the $10,500^{th}$ unit costs approximately \$0.50 extra. In order to increas production from 10,500 units to 11,000 units, 500 additional units need to be made. These additional units will cost a total of $500 \cdot 0.50 =$\$250. Thus, the cost of producing 11,000 units is \$8,200 + \$250 = \$8,450.

4. Evaluate the following.

   (a) (2 points) $\log_2\left(\frac{1}{8}\right)$

   Solution:
   $$\log_2\left(\frac{1}{8}\right) = x \quad \iff \quad 2^x = \frac{1}{8}.$$
   Since $1/8 = 2^{-3}$, we have
   $$2^x = 2^{-3}$$
   and so $x = -3$.

   (b) (2 points) $\log_{64}(8)$

   Solution:
   $$\log_{64}(8) = y \quad \iff \quad 64^y = 8.$$
   Since $8 = 64^{1/2}$, we have
   $$64^y = 64^{1/2}$$
   and so $y = 1/2$.

5. (10 points) Give an equation for the tangent line to the curve

$$x^2 + 4xy + y^2 = 13$$

at the point $(2, 1)$.

Solution: First we perform implicit differentiation to find an expression for $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}[x^2+4xy+y^2] &= \frac{d}{dx}[13] \\ 2x+4y+4x\frac{dy}{dx}+2y\frac{dy}{dx} &= 0 \\ (4x+2y)\frac{dy}{dx} &= -2x-4y \\ \frac{dy}{dx} &= \frac{-2x-4y}{4x+2y} \end{aligned}$$

Since the point $(2,1)$ is on the given curve, plugging in $x=2$ and $y=1$ into the expression above for $\frac{dy}{dx}$ will give the slope of the tangent line to the curve at that point.

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-2(2)-4(1)}{4(2)+2(1)} = \frac{-8}{10} = \frac{-4}{5}$$

The tangent line we seek is the line through $(2,1)$ with slope $-4/5$. An equation for this line is

$$y-1 = \frac{-4}{5}(x-2) \quad \text{or} \quad y = \frac{-4}{5}x+\frac{13}{5}.$$

6. Suppose \$3,000 is invested into an account that earns an annnual interest rate of 6%. What is the value of the investment at the end of 5 years ...
   (a) (4 points) if the interest is compounded monthly?

   Solution:
   $$A(5) = 3000\left(1+\frac{.06}{12}\right)^{12\cdot 5} = 3000(1.005)^{60}$$
   (FYI, this is $\approx \$4,046.55$.)

   (b) (4 points) if the interest is compounded continuously?

   Solution:
   $$A(5) = 3000e^{0.06\cdot 5} = 3000e^{0.3}$$
   (FYI, this is $\approx \$4,049.58$.)

7. (8 points) Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. That is, $C(t)$ obeys the law of natural growth/decay. If the body eliminates half the drug in 30 hours, how long does it take to eliminate 90% of the drug?

Solution: According to the law of natural growth, the concentration $C(t)$ of a drug in the bloodstream can be modelled by function of the form $C(t) = C_0e^{kt}$, where $C(0) = C_0$ is the initial concentration of the drug. Given that $C(30) = C_0/2$, we can solve for either e^k or k.

$$\begin{aligned} C(30) = C_0 e^{30k} &= \frac{C_0}{2} \\ e^{30k} &= \frac{1}{2} \\ e^k &= \left(\frac{1}{2}\right)^{1/30} \quad \text{(solving for } e^k\text{)} \\ k &= \frac{1}{30}\ln\left(\frac{1}{2}\right) \quad \text{(solving for } k\text{)} \end{aligned}$$

Therefore,

$$C(t) = C_0\left(\frac{1}{2}\right)^{t/30} = C_0 e^{\frac{t}{30}\ln\left(\frac{1}{2}\right)}.$$

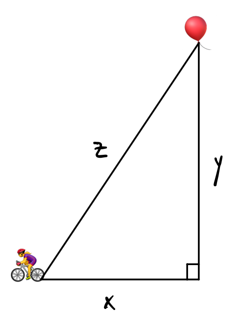
To find the time it takes for the body to eliminate 90% of the drug (so 10% remains), we set $C(t) = C_0/10$ and solve for t.

$$\begin{aligned} C(t) = C_0\left(\frac{1}{2}\right)^{t/30} &= \frac{C_0}{10} \\ \left(\frac{1}{2}\right)^{t/30} &= \frac{1}{10} \\ \frac{t}{30}\ln\left(\frac{1}{2}\right) &= \ln\left(\frac{1}{10}\right) \\ t &= \frac{30\ln(1/10)}{\ln(1/2)} \text{ hours} \end{aligned}$$

(FYI, this is ≈ 99.66 hours.)

8. (10 points) A balloon is rising at a constant speed of 6 ft/s. A girl is cycling along a straight road at a speed of 18 ft/s. When she passes under the balloon, it is 54 ft above her. How fast is the distance between the girl and the balloon increasing 3 seconds later?

Solution:



Let

$x =$ the horizontal distance the girl has travelled since passing under the balloon,
$y =$ the vertical height of the balloon, and
$z =$ the distance between the girl and the balloon.

Using these variables, we are given

$$\frac{dx}{dt} = 18 \text{ ft/s},$$
$$\frac{dy}{dt} = 6 \text{ ft/s},$$

and we are asked to find $\frac{dz}{dt}$. Notice that at all times, x, y, z satisfy the Pythagorean equation

$$x^2 + y^2 = z^2.$$

Taking $\frac{d}{dt}$ of both sides yields

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$
$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}.$$

This shows that the rate at which the distance between the girl and the ballon is increasing depends on $x, y, z, \frac{dx}{dt}$, and $\frac{dy}{dt}$. To find x, we use the fact that the girl has been riding at 18 ft/s for 3 s, and so she has travelled $18 \cdot 3 = 54$ ft since passing under the balloon. To fnd y we use the fact that the balloon had a height of 54 ft when the girl passed underneath, and has been rising at 6 ft/s for 3 s, and so the balloon's height is now $54 + 6 \cdot 3 = 72$ ft. To find z, we use the fact that $x^2 + y^2 = z^2$, and see that $z = \sqrt{54^2 + 72^2}$. Therefore,

$$\frac{dz}{dt} = \frac{(54)(18) + (72)(6)}{\sqrt{54^2 + 72^2}}.$$

(FYI, this is 15.6 ft/s.)

9. (10 points) Find the absolute maximum and minimum values of $f(x) = \frac{3x - 4}{x^2 + 1}$ on the interval $[-2, 2]$.

Solution: We follow the closed interval method. First we find the critical points of f that are in the interval $[-2, 2]$.

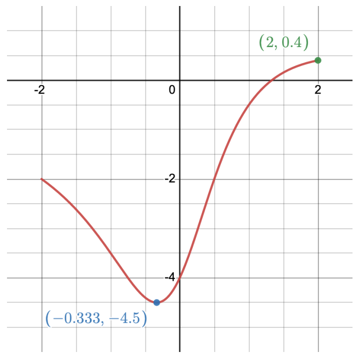
$$f'(x) = \frac{3(x^2 + 1) - 2x(3x - 4)}{(x^2 + 1)^2} = 0$$
$$\frac{-3x^2 + 8x + 3}{(x^2 + 1)^2} = 0$$
$$\frac{-(3x + 1)(x - 3)}{(x^2 + 1)^2} = 0$$
$$-(3x + 1)(x - 3) = 0$$
$$x = \frac{-1}{3}, 3$$

Note that the derivative $f'(x)$ is never undefined, so these are the only critical points of f. And the only critical point in the interval $[-2, 2]$ is $x = -1/3$.

Now we evaluate f at all critical numbers in the interval $[-2, 2]$ and at the endpoints $x = -2$ and $x = 2$.

x	$f(x)$
-2	-2
-1/3	-9/2
2	2/5

Therefore, f has absolute minimum value $-9/2$ and absolute maximum value $2/5$.



10. Consider the function $f(x) = 3x^5 - 5x^3 + 3$.
    (a) (6 points) Find the intervals on which f is increasing/decreasing.

Solution: In order to find where f is increasing/decreasing, we find where the derivative f' is positive/negative. Since the sign of f' can only change at critical numbers, the critical numbers cut the domain of f into intervals on which the sign of f' is constant. We can determine the sign of f' on each interval by determining the sign of each fator of f' on each interval, organizing our work with a sign table.
The derivative is

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1).$$

The derivative is a polynomial and so is defined for all real numbers. The only critical numbers for f are the solutions to $f'(x) = 0$. That is,

$$f'(x) = 15x^2(x+1)(x-1) = 0$$
$$x = -1, 0, \text{ or } 1.$$

	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
$15x^2$	pos	pos	pos	pos
$x+1$	neg	pos	pos	pos
$x-1$	neg	neg	neg	pos
$f'(x)$	pos	neg	neg	pos
$f(x)$	incr	decr	decr	incr

Thus $f(x)$ is increasing on $(-\infty,-1) \cup (1,\infty)$ and decreasing on $(-1,0) \cup (0,1)$, or you could say f is decreasing on $(-1,1)$.

(b) (2 points) Find the local maximum and minimum values of f.

Solution: Since f switched from increasing to decreasing at $x = -1$,

$$f(-1) = 3(-1)^5 - 5(-1)^3 + 3 = 5$$

is a local maximum value. Since f switches from decreasing to increasing at $x = 1$,

$$f(1) = 3(1)^5 - 5(1)^3 + 3 = 1$$

is a local minumum value.

(c) (6 points) Find the intervals of concavity and the inflection points.

Solution: In order to find where f is concave up/down, we find where the second derivative f'' is positive/negative. This is just like what we did in part (a) but for f' instead of f, because determining where f is convave up/down is the same thing as determining where f' is increasing/decreasing.

$$\begin{aligned} f''(x) = 60x^3 - 30x &= 0 \\ 30x(2x^2 - 1) &= 0 \\ 30x(\sqrt{2}x + 1)(\sqrt{2}x - 1) &= 0 \\ x &= 0, \frac{\pm 1}{\sqrt{2}} \end{aligned}$$

	$(-\infty, -1/\sqrt{2})$	$(-1/\sqrt{2}, 0)$	$(0, 1/\sqrt{2})$	$(1/\sqrt{2}, \infty)$
$30x$	neg	neg	pos	pos
$\sqrt{2}x + 1$	neg	pos	pos	pos
$\sqrt{2}x - 1$	neg	neg	neg	pos
$f''(x)$	neg	pos	neg	pos
$f'(x)$	decr	incr	decr	incr
$f(x)$	C.D.	C.U.	C.D.	C.U.

Thus f is concave up on $(-1/\sqrt{2}, 0) \cup (1/\sqrt{2}, \infty)$ and concave down on $(-\infty, -1/\sqrt{2}) \cup (0, 1/\sqrt{2})$. Since f changes concavity at $x = 0, \pm 1/\sqrt{2}$, the graph of f has three inflection points

$$\left(\frac{-1}{\sqrt{2}}, f\left(\frac{-1}{\sqrt{2}}\right)\right), (0, f(0)), \text{ and } \left(\frac{1}{\sqrt{2}}, f\left(\frac{1}{\sqrt{2}}\right)\right).$$

That is,

$$\left(\frac{-1}{\sqrt{2}}, \frac{7 + 12\sqrt{2}}{4\sqrt{2}}\right), (0, 3), \text{ and } \left(\frac{1}{\sqrt{2}}, \frac{-7 + 12\sqrt{2}}{4\sqrt{2}}\right).$$