## Exam 2 Practice Problems

1. Find $\frac{d y}{d x}$.
(a) $y=50\left(\frac{1}{2}\right)^{x / 9}$

Solution: Let $y=50\left(\frac{1}{2}\right)^{u}$ and $u=x / 9$. Then

$$
\frac{d y}{d x}=\frac{d y}{d u} \cdot \frac{d u}{d x}=50\left(\frac{1}{2}\right)^{u} \ln \left(\frac{1}{2}\right) \cdot \frac{1}{9}
$$

Since $\ln (1 / 2)=\ln 1-\ln 2=0-\ln 2$, this can be simplified to

$$
\frac{d y}{d x}=\frac{-50 \ln 2}{9}\left(\frac{1}{2}\right)^{x / 9}
$$

(b) $y=\frac{e^{3 x}}{\left(1+x^{2}\right)^{2}}$

## Solution:

$$
\begin{aligned}
\frac{d y}{d x} & =\frac{\frac{d}{d x}\left[e^{3 x}\right]\left(1+x^{2}\right)^{2}-e^{3 x} \frac{d}{d x}\left[\left(1+x^{2}\right)^{2}\right]}{\left(\left(1+x^{2}\right)^{2}\right)^{2}} \\
& =\frac{e^{3 x} \cdot 3\left(1+x^{2}\right)^{2}-e^{3 x} \cdot 2\left(1+x^{2}\right) 2 x}{\left(1+x^{2}\right)^{4}} \\
& =\frac{e^{3 x}\left(3 x^{2}-4 x+3\right)}{\left(1+x^{2}\right)^{3}}
\end{aligned} \quad \text { (unsimplified) }
$$

For a question like this, you do not have to simplify your answer. However, if we were searching for critical points or needed to find the second derivative, then simplifying would make those things easier.
(c) $y=2^{5 x} \ln \left(1-x^{2}\right)$

## Solution:

$$
\begin{array}{rlr}
\frac{d y}{d x} & =\frac{d}{d x}\left[2^{5 x}\right] \ln \left(1-x^{2}\right)+2^{5 x} \frac{d}{d x}\left[\ln \left(1-x^{2}\right)\right] & \\
& =2^{5 x} \cdot 5 \ln (2) \ln \left(1-x^{2}\right)+2^{5 x} \cdot \frac{-2 x}{1-x^{2}} & \text { (unsimplified) } \\
& =2^{5 x} \ln (32) \ln \left(1-x^{2}\right)-\frac{x 2^{5 x+1}}{1-x^{2}} & \text { (simplified) }
\end{array}
$$

In the last line I'm just playing with different ways to write exponential and logarithmic expressions. Note that $2 \cdot 2^{5 x}=2^{5 x+1}$ and $5 \ln (2)=\ln \left(2^{5}\right)=\ln (32)$.
(d) $y=\ln \left(\sqrt[3]{\frac{e^{2}}{x(2 x+7)^{4}}}\right)$

Solution: Before taking the derivative, let us apply log laws to break up the function into simple pieces.

$$
\begin{aligned}
\ln \left(\sqrt[3]{\frac{e^{2}}{x(2 x+7)^{4}}}\right) & =\frac{1}{3}\left(\ln \left(e^{2}\right)-\ln \left(x(2 x+7)^{4}\right)\right) \\
& =\frac{1}{3}(2-(\ln (x)+4 \ln (2 x+7))) \\
& =\frac{1}{3}(2-\ln (x)-4 \ln (2 x+7))
\end{aligned}
$$

Now

$$
\begin{aligned}
\frac{d y}{d x} & =\frac{d}{d x}\left[\frac{1}{3}(2-\ln (x)-4 \ln (2 x+7))\right] \\
& =\frac{-1}{3}\left(\frac{1}{x}-\frac{8}{2 x+7}\right)
\end{aligned}
$$

Notice that applying log laws allowed us to avoid complicated derivatives. Remember that there usually several different ways to write logarithmic expressions. Choose the way that makes the work easiest.
2. Suppose $C(q)$ is the cost that a company must pay to produce $q$ units. If $C(1,250)=3,600$ and $C^{\prime}(1,250)=2.4$, approximately how much would it cost the company to produce 1,300 units?

Solution: We are given that it costs $\$ 3,600$ to produce 1,250 units, and each additional unit after the $1,250^{t h}$ unit costs approximately $\$ 2.40$ extra. In order to increas production from 1,250 units to 1,300 units, 50 additional units need to be made. These additional units will cost a total of $50 \cdot 2.4=\$ 120$. Thus, the cost of producing 1,300 units is $\$ 3,600+\$ 120=\$ 3,720$.
3. A company has the following cost and demand functions.

$$
C(q)=84+1.26 q-.01 q^{2}+.00007 q^{3}, \quad p=3.5-.01 q
$$

(a) If the price of each unit is $\$ 1.20$, how many units will be sold?

Solution: This is example 5 on page 154 of Brief Applied Calculus by Stewart and Clegg.
(b) Determine the production level that will maximize profit for the company.

Solution: This is example 5 on page 154 of Brief Applied Calculus by Stewart and Clegg.
4. Evaluate $\log _{4}\left(\frac{1}{16}\right)$ and $\log _{9}(3)$.

## Solution:

$$
\log _{4}\left(\frac{1}{16}\right)=x \quad \Longleftrightarrow \quad 4^{x}=\frac{1}{16} .
$$

Since $1 / 16=4^{-2}$, we have

$$
4^{x}=4^{-2}
$$

and so $x=-2$.

$$
\log _{9}(3)=y \quad \Longleftrightarrow \quad 9^{y}=3
$$

Since $3=9^{1 / 2}$, we have

$$
9^{y}=9^{1 / 2}
$$

and so $y=1 / 2$.
5. Give an equation for the tangent line to the curve

$$
2\left(x^{2}+y^{2}\right)^{2}=25\left(x^{2}-y^{2}\right)
$$

at the point $(3,1)$.

Solution: First we perform implicit differentiation to find an expression for $\frac{d y}{d x}$.

$$
\begin{aligned}
4\left(x^{2}+y^{2}\right)\left(2 x+2 y \frac{d y}{d x}\right) & =25\left(2 x-2 y \frac{d y}{d x}\right) \\
8 x^{3}+8 x^{2} y \frac{d y}{d x}+8 x y^{2}+8 y^{3} \frac{d y}{d x} & =50 x-50 y \frac{d y}{d x} \\
\left(8 x^{2} y+8 y^{3}+50 y\right) \frac{d y}{d x} & =50 x-8 x^{3}-8 x y^{2} \\
\frac{d y}{d x} & =\frac{50 x-8 x^{3}-8 x y^{2}}{8 x^{2} y+8 y^{3}+50 y}
\end{aligned}
$$

Since the point $(3,1)$ is on the given curve, plugging in $x=3$ and $y=1$ into the expression above for $\frac{d y}{d x}$ will give the slope of the tangent line to the curve at that point.

$$
\left.\frac{d y}{d x}\right|_{(3,1)}=\frac{50(3)-8(3)^{3}-8(3)(1)^{2}}{8(3)^{2}(1)+8(1)^{3}+50(1)}=\frac{-9}{13}
$$

The tangent line we seek is the line through $(3,1)$ with slope $-9 / 13$. An equation for this line is

$$
y-1=\frac{-9}{13}(x-3) \quad \text { or } \quad y=\frac{-9}{13} x+\frac{40}{13}
$$

6. How much money would need to be deposited into an account that earns $6 \%$ annual interest compounded quarterly so that it is worth $\$ 15,000$ in 5 years?

Solution: Solve for $P$.

$$
\begin{aligned}
A(5)=P\left(1+\frac{.06}{4}\right)^{4 \cdot 5} & =15,000 \\
P & =\frac{15,000}{1.015^{20}} \text { dollars }
\end{aligned}
$$

FYI, this is approximately $\$ 11,137.06$. But you will not have a calculator when taking the exam, so you would leave your answer as the expression above.
7. How long will it take an investment to double if it earns $5 \%$ annual interest compounded continuously?

Solution: Solve for $t$.

$$
\begin{aligned}
A(t)=P e^{.05 t} & =2 P \\
e^{.05 t} & =2 \\
.05 t & =\ln (2) \\
t & =\frac{\ln (2)}{.05}=20 \ln (2) \text { years }
\end{aligned}
$$

FYI, this is approximately 13.86 years.
8. Suppose a radioactive material takes 3 years to decay to $99 \%$ of its original mass. Find the half-life of this material. Assume that the mass of the radioactive material obeys the law of natural growth.

Solution: The law of natural growth allows us to model the mass of the material after $t$ years with the function

$$
M(t)=C e^{k t}
$$

where $C$ is the initial mass of the material. We are given that $M(3)=.99 C$, that is

$$
\begin{aligned}
C e^{3 k} & =.99 C \\
e^{3 k} & =.99 \\
e^{k} & =.99^{1 / 3}
\end{aligned}
$$

Therefore the mass after $t$ years is given by

$$
M(t)=C(.99)^{t / 3}
$$

Note that we could have solved for the (negative) relative growth rate $k$ above and written this same function as

$$
M(t)=C e^{\frac{1}{3} \ln (.99) t} \quad\left(k=\frac{1}{3} \ln (.99)\right) .
$$

To find the half-life, we set $M(t)=C / 2$ and solve for $t$.

$$
\begin{aligned}
C(.99)^{t / 3} & =\frac{C}{2} \\
(.99)^{t / 3} & =\frac{1}{2} \\
\frac{t}{3} \ln (.99) & =\ln (1 / 2) \\
t & =\frac{3 \ln (1 / 2)}{\ln (.99)} \text { years }
\end{aligned}
$$

FYI, this is approximately 206.9 years.
9. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1 \mathrm{ft} / \mathrm{s}$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

## Solution:



Let $x$ be the horizontal distance from the wall to the base of the ladder. Let $y$ be the vertical distance from the ground to the top of the ladder. Since we are given the length of the ladder is 10 ft , we have

$$
\begin{equation*}
x^{2}+y^{2}=10 \tag{1}
\end{equation*}
$$

Differentiating both sides with respect to $x$, we have

$$
\begin{aligned}
2 x \frac{d x}{d t}+2 y \frac{d y}{d t} & =0 \\
\frac{d y}{d t} & =\frac{-x \frac{d x}{d t}}{y}
\end{aligned}
$$

Since we are given that $x=6$, equation (1) implies $y=8$. And since we are given that $\frac{d x}{d t}=1$, we now have

$$
\frac{d y}{d t}=\frac{-6(1)}{8}=\frac{-3}{4} \mathrm{ft} / \mathrm{s}
$$

That is, the top of the ladder sliding down the wall at a rate of $3 / 4 \mathrm{ft} / \mathrm{s}$.
10. Find the absolute maximum and absolute minimum values of $f(x)=\frac{x}{x^{2}+4}$ on the interval $[0,3]$.

Solution: We follow the closed interval method. First we find the critical points of $f$ that are in the interval $[0,3]$.

$$
\begin{aligned}
f^{\prime}(x)=\frac{4-x^{2}}{\left(x^{2}+4\right)} & =0 \text { or undefined } \\
4-x^{2} & =0 \quad\left(f^{\prime}(x) \text { is never undefined }\right) \\
(2+x)(2-x) & =0 \\
x & =-2,2
\end{aligned}
$$

The only critical point in the interval $[0,3]$ is $x=2$.
Now we evaluate $f$ at all critical numbers in the interval $[0,3]$ and at the endpoints $x=0$ and $x=3$.

| $x$ | $f(x)$ |
| :---: | :---: |
| 0 | 0 |
| 2 | $1 / 4$ |
| 3 | $3 / 13$ |

Therefore, $f$ has absolute minimum value 0 . Since

$$
\frac{1}{4}=\frac{3}{12}>\frac{3}{13}
$$

$f$ has absolute maximum value $1 / 4$.
11. Consider the function $f(x)=x^{4}-2 x^{2}+2$.
(a) Find the intervals on which $f$ is increasing/decreasing.

Solution: In order to find where $f$ is increasing/decreasing, we find where the derivative $f^{\prime}$ is positive/negative. Since the sign of $f^{\prime}$ can only change at critical numbers, the critical numbers cut the domain of $f$ into intervals on which the sign of $f^{\prime}$ is constant. We can determine the sign of $f^{\prime}$ on each interval by determining the sign of each fator of $f^{\prime}$ on each interval, organizing our work with a sign table.

|  | $\begin{aligned} & f^{\prime}(x)=4 x^{3}-4 x=0 \\ & 4 x(x+1)(x-1)=0 \end{aligned}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |
|  | $x=-1,0,1$ |  |  |  |  |  |
|  | $(-\infty,-1)$ |  | $(-1,0)$ | ; $(0,1)$ | , | $(1, \infty)$ |
| $4 x$ | neg | I | neg | pos | I | pos |
| $x+1$ | neg |  | pos | , pos | , | pos |
| $x-1$ | neg |  | neg | neg | 1 | pos |
| $f^{\prime}(x)$ | neg | 1 | pos | neg | 1 | pos |
| $f(x)$ | decr | , | incr | ) decr | 1 | incr |

Thus $f(x)$ is increasing on $(-1,0) \cup(1, \infty)$ and decreasing on $(-\infty,-1) \cup(0,1)$.
(b) Find the local maximum and minimum values of $f$.

Solution: We can now use the First Derivative Test to find the local maximum and minuimum values of $f$. Since $f$ changes from increasing to decreasing at $x=0, f(0)=2$ is a local maximum value. Since $f$ changes from decreasing to increasing at $x=-1$ and at $x=1$, both $f(-1)$ and $f(1)$ are local minimum values. In this case they are the same, so $f$ has only one local minimum value $f( \pm 1)=1$.
(c) Find the intervals of concavity and the inflection points.

Solution: In order to find where $f$ is concave up/down, we find where the second derivative $f^{\prime \prime}$ is positive/negative.

$$
\begin{aligned}
f^{\prime \prime}(x)=12 x^{2}-4 & =0 \\
4\left(3 x^{2}-1\right) & =0 \\
4(\sqrt{3} x+1)(\sqrt{3} x-1) & =0 \\
x & =\frac{ \pm 1}{\sqrt{3}}
\end{aligned}
$$



Thus $f$ is concave up on $(-\infty,-1 / \sqrt{3}) \cup(1 / \sqrt{3}, \infty)$ and concave down on $(-1 / \sqrt{3}, 1 / \sqrt{3})$. Since $f$ changes from cancave up to cancave down at $x=-1 / \sqrt{3}$ and from concave down to concave up at $x=1 / \sqrt{3}$, the graph of $f$ has inflection points at $(-1 / \sqrt{3}, f(-1 / \sqrt{3}))=(-1 / \sqrt{3}, 13 / 9)$ and $(1 / \sqrt{3}, f(1 / \sqrt{3}))=(1 / \sqrt{3}, 13 / 9)$.

