Exam 2 Study Guide

Abstract

Exam 2 will cover sections 3.1-6 and 4.1-3 of our textbook, *Brief Applied Calculus* by Stewart and Clegg. This is just my own summary. This document is intended to supplement (not replace) the reading of those sections of the textbook.

Contents

1 Definitions

You should know the following definitions.

Definition 1. Let $C(q)$ be a **cost function** that gives the cost in dollars of producing q units. Then the **average cost** per unit when producing q units is $C(q)/q$, and the **marginal cost** when producing q units is $C'(q)$. The marginal cost $C'(q)$ is a very good approximation for the additional cost of producing each additional unit after the q^{th} unit.

Definition 2. (Very similar to definition 1) Let $R(q)$ be a **revenue function** that gives the revenue (money received) in dollars when producing (and selling) q units. The **average revenue** per unit when producing q units is $R(q)/q$, and the **marginal revienue** is $R'(q)$. The marginal revenue $R'(q)$ is a very good approximation for the additional revenue earned for each additional unit produced and sold after the q^{th} unit.

Definition 3. Let $p = D(q)$ be the price per unit that a company can charge if it sells q units. Then D is called the **demand function** (or **price function**) and its graph is called a demand curve. Since revenue is the number of units sold times the price of each unit, the revenue function is $R(q) = qp = q \cdot D(q)$.

Definition 4. The **profit function** $P(q)$ is formed by subtracting the total cost from the total revenue:

$$P(q) = R(q) - C(q).$$

Definition 5. The **logarithmic function with base** a is defined as follows.

$$\log_a w = z \quad \iff \quad a^z = w$$

Definition 6. The **natural logarithm** $\ln x$ is the logarithmic function with base $e \approx 2.718$.

$$\ln x = \log_e x$$

Definition 7. Any quantity that increases or decreases over time according the **law of natural growth** can be modeled by a function of the form

$$A(t) = Ce^{kt},$$

where $C = A(0)$ is the initial amount, and k is the **relative growth rate**.

Definition 8. When a quantity *increases* according to the law of natural growth, it is said to experience **exponential growth**. When a quantity *decreases* according to the law of natural growth, it is said to experience **exponential decay**. The amount of time it takes a quantity that experiences exponential decay to decrease to half its initial amount is called its **half-life**, typically denoted λ.

Definition 9. The number c is a **critical number** (also **critical point**) of f if c is in the domain of f and either $f'(c) = 0$ or $f'(c)$ does not exist.

Definition 10. A function f has **absolute maximum value** $f(c)$ if $f(c) \geq f(x)$ for all x in the domain of f. A function f has **absolute minimum value** $f(c)$ if $f(c) \leq f(x)$ for all x in the domain of f.

Definition 11. A function f has **local maximum value** $f(c)$ if $f(c) \geq f(x)$ for all x in a "neighborhood" of c (i.e. an open interval that contains c). A function f has **local minimum value** $f(c)$ if $f(c) \leq f(x)$ for all x in a "neighborhood" of c Note that a function *cannot* attain a *local* maximum/minimum value at an endpoint of its domain.

Definition 12. A function f is **increasing** on an interval if $f'(x) > 0$ on that interval. A function f is **decreasing** on an interval if $f'(x) < 0$ on that interval.

Definition 13. A function f is **concave up** on an interval if $f''(x) > 0$ on that interval. A function f is **concave down** on an interval if $f''(x) < 0$ on that interval. The graph $y = f(x)$ has an **inflection point** at $(c, f(c))$ if f changes from concave up to concave down or vice versa at $x = c$.

2 Techniques/concepts

1. Cost $C(q)$ is minimized when the production level q is such that the average cost is equal to then marginal cost:
$$\frac{C(q)}{q} = C'(q).$$

2. Profit $P(q)$ is maximized when the production level is such that the marginal revenue is equal to the marginal cost:
$$R'(q) = C'(q).$$

3. Log properties (these follow from definitions 5 and 6):
   (a) $\log_a(1) = 0$ and in particular $\ln(1) = 0$
   (b) $\log_a(a) = 1$ and in particular $\ln(e) = 1$
   (c) $\log_a(a^x) = x$ and in particular $\ln(e^x) = x$
   (d) $e^{\log_a(x)} = x$ when $x > 0$ and in particular $e^{\ln(x)} = x$ when $x > 0$
4. Log laws (these are the laws of exponents restated using logarithms):
   (a) $\ln(AB) = \ln(A) + \ln(B)$
   (b) $\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$
   (c) $\ln(A^c) = c\ln(A)$
5. A principal investment of P dollars that earns annual interest rate r (converted from percent to decimal) compounded n times per year is worth

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

   after t years. If interest is compounded continuously, then it is worth

$$A(t) = Pe^{rt}$$

   after t years.
6. Implicit differentiation
   (a) Take the derivative $\frac{d}{dx}$ of both sides of *any* equation relating x and y.
   (b) Remember to treat y as a function of f with derivative $\frac{dy}{dx}$ and use the chain rule.
7. Related rates
   (a) Draw a picture. Look for nice geometric shapes like triangles.
   (b) Label all of the lengths/distances in your picture. Some or all of these distances change and depend on time – these distances should be labelled with *variables*. Only distances that do not ever change can be labelled with numbers (at this stage).
   (c) Using your chosen variables, write down any/all known rates of change in terms of derivatives. Also write down the rate of change we must find as a derivative.
   (d) Write an equation that relates the variables whose derivatives we wrote down in step 7c. Use your picture and a little bit of geometry. If you find yourself wanting to use additional variables, try to find a way to express them in terms of only the variables whose derivatives we wrote down in step 7c.
   (e) Take the derivative $\frac{d}{dt}$ of both sides of your equation. Remember to treat all variables as functions of t and use the chain rule.
   (f) Now solve for the unknown rate of change and plug in values for all other rates of change (given) and all variables (also given, though a side calculation may be needed to find certain values).

8. The Closed Interval Method for finding absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$
   (a) Find the critical numbers of f in the interval (a, b).
   (b) Find the values of f at the critical numbers in (a, b) as well as the endpoints a and b.
   (c) The largest value is the absolute maximum value and the smallest is the absolute minimum vaue of f.

9. Sign tables - know how to use them to determine on which intervals a function (like f' or f'') is positive/negative. This is how we find intervals on which a function is increasing/decreasing and/or concave up/down. For a better explanation, see Practice Problem 11 and/or examples 1, 4, and 5 in section 4.3 of the textbook.

10. First derivative test for finding local maximum and minimum values:
    (a) If $f'(c) = 0$ and f' changes from positive to negative (f changes from increasing to decreasing) at $x = c$ then $f(c)$ is a local maximum value of f.
    (b) If $f'(c) = 0$ and f' changes from negative to positive (f changes from decreasing to increasing) at $x = c$ then $f(c)$ is a local minimum value of f.

    Note that in order to use the first derivative test, we first need to know on which intervals f is increasing/decreasing, which usually requires the use of a sign table.

11. Second derivative test for finding local maximum and minimum values:
    (a) If $f'(c) = 0$ and $f''(x) < 0$ (concave down) then $f(c)$ is a local maximum value of f.
    (b) If $f'(c) = 0$ and $f''(x) > 0$ (cancave up) then $f(c)$ is a local minimum value of f.
    (c) If $f'(c) = 0$ and $f''(x) = 0$ then it is uncertain whether $f(c)$ is a local maximum, minimum, or neither. In this case, the first derivative test *must* be used.

3 Finding derivatives

You should have memorized the following differentiation rules and be able to use them.

	Function	Derivative
Constant function	c	0
Power Rule	x^n	nx^{n-1}
Power Rule with Chain Rule	$[g(x)]^n$	$n[g(x)]^{n-1}g'(x)$
Reciprocal function	$\frac{1}{x}$	$-\frac{1}{x^2}$
Reciprocal function with Chain Rule	$\frac{1}{g(x)}$	$-\frac{g'(x)}{[g(x)]^2}$
Natural exponential function	e^x	e^x
Natural exponential function with Chain Rule	$e^{g(x)}$	$e^{g(x)}g'(x)$
Exponential function	a^x	$a^x \ln a$
Natural logarithmic function	$\ln x$	$1/x$
Natural logarithmic function with Chain Rule	$\ln g(x)$	$\frac{g'(x)}{g(x)}$
Constant Multiple Rule	$cf(x)$	$cf'(x)$
Sum Rule	$f(x) + g(x)$	$f'(x) + g'(x)$
Difference Rule	$f(x) - g(x)$	$f'(x) - g'(x)$
Product Rule	$f(x)\,g(x)$	$f(x)\,g'(x) + g(x)\,f'(x)$
Quotient Rule	$\frac{f(x)}{g(x)}$	$\frac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$
Chain Rule	$f(g(x))$	$f'(g(x))\,g'(x)$
Chain Rule (Leibniz notation)	$y = f(u),\ u = g(x)$	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Figure 1: This table comes from page 186 of *Brief Applied Calculus* by Stewart and Clegg.