Quiz 4

Name: *Answer Key* Section: ______

Answer all 5 questions for a total of 100 points. Write your solutions in the space provided and put a box around your final answers. Answers can be left as logarithmic/exponential expressions, or a calculator can be used to write your answers numeriacally.

1. (25 points) The equation

$$x^2 - xy + y^2 = 3$$

represents a "rotated ellipse", that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis (i.e. x-intercepts) and show that the tangent lines at these points are parallel (i.e. have the same slope).

THE ELLIPSE CROSSES THE X-AXIS WHEN $y = 0$:

$$x^2 - x(0) + (0)^2 = 3 \implies x^2 = 3$$

$$x = \pm\sqrt{3} \implies \text{AT POINTS } \boxed{(\pm\sqrt{3}, 0).}$$

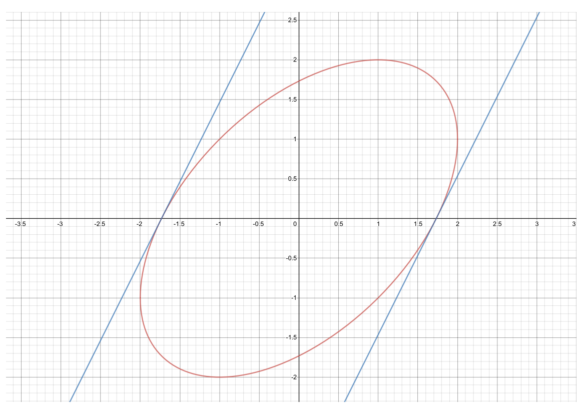
FIND $\frac{dy}{dx}$: $\frac{d}{dx}\left[x^2 - xy + y^2\right] = \frac{d}{dx}[3]$

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{y - 2x}{2y - x}}$$

$$\left.\frac{dy}{dx}\right|_{(\sqrt{3},0)} = \frac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$$

$$\left.\frac{dy}{dx}\right|_{(-\sqrt{3},0)} = \frac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$$



$\therefore$ THE TANGENT LINES TO THE CURVE AT $(\pm\sqrt{3}, 0)$ HAVE THE SAME SLOPE, 2 $\implies$ PARALLEL.

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2. (10 points) Find the derivative of $f(x) = \ln\left(\sqrt{\dfrac{3x+2}{3x-2}}\right)$.

   Hint: the calculus will be much easier if you first apply log laws to $f(x)$.

$$f(x) = \ln\left(\left(\frac{3x+2}{3x-2}\right)^{1/2}\right) = \frac{1}{2}\ln\left(\frac{3x+2}{3x-2}\right)$$

$$= \frac{1}{2}\left(\ln(3x+2) - \ln(3x-2)\right)$$

$$f'(x) = \frac{1}{2}\left(\frac{1}{3x+2}\cdot 3 - \frac{1}{3x-2}\cdot 3\right) = \boxed{\frac{3}{2}\left(\frac{1}{3x+2} - \frac{1}{3x-2}\right)}$$

3. (20 points) Suppose a sample of radioactive material has an initial mass of 92.3 grams and decays exponentially. If its mass 10 days later is 91.8 grams, find the half-life of the material.

DECAYS EXPONENTIALLY : $M(t) = Ce^{kt}$

GIVEN (1) $M(0) = 92.3$, so $92.3 = C\underbrace{e^{k\cdot 0}}_{1} \Rightarrow C = 92.3$

(2) $M(10) = 91.8$, so $91.8 = 92.3e^{k\cdot 10} \Rightarrow \dfrac{91.8}{92.3} = e^{k\cdot 10}$

$$\Rightarrow \left(\frac{91.8}{92.3}\right)^{1/10} = e^{k} \quad \text{AND} \quad k = \ln\left(\left(\frac{91.8}{92.3}\right)^{1/10}\right)$$

$$\therefore\ M(t) = 92.3\left(\frac{91.8}{92.3}\right)^{t/10} = 92.3\, e^{\frac{1}{10}\ln\left(\frac{91.8}{92.2}\right)t} \qquad (\approx -.000543)$$

NOW SOLVE : $\cancel{92.3}\left(\dfrac{91.8}{92.3}\right)^{t/10} = \underset{\text{"HALF"}}{\dfrac{1}{2}} \cdot \cancel{92.3} \Rightarrow \left(\dfrac{91.8}{92.3}\right)^{t/10} = \dfrac{1}{2}$

$$\Rightarrow \frac{t}{10}\ln\left(\frac{91.8}{92.3}\right) = \ln\left(\frac{1}{2}\right) \Rightarrow t = \boxed{\frac{10\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{91.8}{92.3}\right)} \text{ DAYS } (\approx 1{,}276)}$$

(DON'T NEED TO INCLUDE UNITS)

4. How long does it take an investment to double if it earns 4.68% annual interest…

(a) (10 points) compounded semi-annually (twice per year)?

COMPOUND INTEREST (n TIMES PER YEAR): $A(t) = P\left(1+\frac{r}{n}\right)^{nt} = P\left(1+\frac{.0468}{2}\right)^{2t} = P(1.0234)^{2t}$

SOLVE FOR t: $A(t) = 2P$ (DOUBLE)

$$\cancel{P}(1.0234)^{2t} = 2\cancel{P}$$

$$2t\ln(1.0234) = \ln(2) \Rightarrow \boxed{t = \frac{\ln(2)}{2\ln(1.0234)} \text{ YEARS}}$$

≈ 15 YEARS

(b) (10 points) compounded continuously?

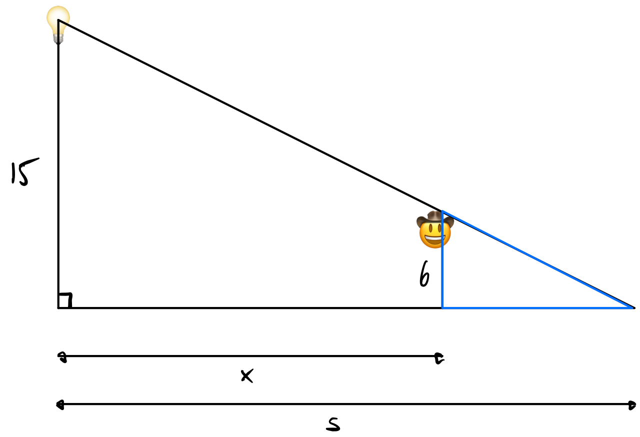
CONTINUOUSLY COMPOUNDED INTEREST: $A(t) = Pe^{rt} = Pe^{.0468t}$

SOLVE FOR t: $\cancel{P}e^{.0468t} = 2\cancel{P}$ (DOUBLE)

$$.0468t = \ln(2) \Rightarrow \boxed{t = \frac{\ln(2)}{.0468} \text{ YEARS}}$$

OR ≈ 14.8 YEARS

5. (25 points) A street light is mounted at the top of a pole 15 ft tall. A man 6 ft tall walks away from the pole with speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

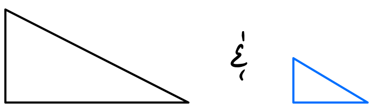


LET x = DISTANCE FROM POLE TO MAN

s = DISTANCE FROM POLE TO TIP OF SHADOW

GIVEN: $\frac{dx}{dt} = 5$ ft/s

FIND: $\frac{ds}{dt}$



ARE SIMILAR TRIANGLES.

THEREFORE, $\frac{\text{BASE}}{\text{HEIGHT}} = \frac{s}{15} = \frac{s-x}{6}$

$$\Rightarrow \quad 6s = 15(s-x) = 15s - 15x$$

$$15x = 9s$$

$$\boxed{5x = 3s}$$ (RELATED QUANTITIES)

$$\frac{d}{dt}\left[5x\right] = \frac{d}{dt}\left[3s\right] \quad \Rightarrow \quad 5\frac{dx}{dt} = 3\frac{ds}{dt}$$

$$\therefore \quad \frac{ds}{dt} = \frac{5}{3}\frac{dx}{dt} = \frac{5}{3}(5 \text{ ft/s}) = \boxed{\frac{25}{3} \text{ ft/s}}$$

(NOTE THAT $\frac{ds}{dt}$ DOES NOT DEPEND ON x, IT IS CONSTANT.

THE TIP OF THE SHADOW MOVES AT CONSTANT SPEED.)