Quiz 5 Solutions

Name: ______________________________ Section: __________

> Answer questions 1-3 for a total of 100 points. Answer question 4 for 20 additional bonus points. Write your solutions in the space provided and put a box around your final answers.

1. Find the absolute maximum and absolute minimum values of f on the given interval.

   (a) (20 points) $f(x) = x^3 - 6x^2 + 5$, $-3 \leq x \leq 5$

Solution: (Closed interval method, §4.2) First we find the critical numbers of f, i.e. points c in the domain such that $f'(c)$ is zero or undefined. Since $f'(x) = 3x^2 - 12x$ is a polynomial, it is never undefined. So we solve $f'(x) = 0$.

$$\begin{aligned} f'(x) = 3x^2 - 12x &= 0 \\ 3x(x-4) &= 0 \\ x &= 0, 4. \end{aligned}$$

Next, since all critical numbers are in the specified domain $[-3, 5]$, we evaluate f at all critical numbers in the closed interval $[-3, 5]$ as well as at the endpoints.

x	$f(x)$
-3	-58
0	5
4	-79
5	-130

Therefore f has absolute maximum value $f(0) = 5$ and absolute minimum value $f(5) = -130$.

   (b) (20 points) $g(x) = x - \sqrt[3]{x}$, $-1 \leq x \leq 4$

Solution: (Closed interval method, §4.2) First we find the critical numbers of g.

$$\begin{aligned} g'(x) = 1 - \frac{1}{3}x^{-2/3} &= 0 \\ x^{-2/3} &= 3 \\ x^{-1/3} &= \pm\sqrt{3} \\ x &= \pm\sqrt{3}^{-3} = \pm\frac{1}{3\sqrt{3}} \end{aligned}$$

Next, since all critical numbers are in the specified domain $[-1, 4]$, we evaluate g at all critical numbers in the closed interval $[-1, 4]$ as well as at the endpoints.

x	$g(x)$
-1	0
$-1/3\sqrt{3}$	$2/3\sqrt{3} \approx .3849$
$1/3\sqrt{3}$	$-2/3\sqrt{3} \approx -.3849$
4	$4 - 4^{1/3} \approx 2.4126$

Therefore g has absolute maximum value $g(4) = 4 - 4^{1/3} \approx 2.4126$ and absolute minimum value $g(1/3\sqrt{3}) = -2/3\sqrt{3} \approx -.3849$.

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2. Consider the function

$$f(x) = (x+1)^5 - 5x - 2.$$

(a) (20 points) Find the intervals on which f is increasing/decreasing and all local maximum/minimum values of f.

Solution: In order to find where f is increasing/decreasing, we find where the derivative f' is positive/negative. Since the sign of f' can only change at critical numbers, the critical numbers cut the domain of f into intervals on which the sign of f' is constant. We can determine the sign of f' on each interval by determining the sign of each fator of f' on each interval, organizing our work with a sign table.

$$\begin{aligned}
f'(x) = 5(x+1)^4 - 5 &= 0\\
5[(x+1)^4 - 1] &= 0\\
5((x+1)^2+1)((x+1)^2-1) &= 0 \text{ (difference of squares)}\\
5((x+1)^2+1)((x+1)+1)((x+1)-1) &= 0 \text{ (difference of squares)}\\
5x((x+1)^2+1)(x+2) &= 0\\
x &= -2, 0
\end{aligned}$$

	$(-\infty,-2)$	$(-2,0)$	$(0,\infty)$
$5x$	neg	neg	pos
$((x+1)^2+1)$	pos	pos	pos
$(x+2)$	neg	pos	pos
$f'(x)$	pos	neg	pos
$f(x)$	incr	decr	incr

Thus $f(x)$ is increasing on $(-\infty,-2)\cup(0,\infty)$ and decreasing on $(-2,0)$. We can now use the First Derivative Test (§4.3) to find the local maximum and minuimum values of f. Since f changes from increasing to decreasing at $x=-2$, $f(-2)=7$ is a local maximum value. Since f changes from decreasing to increasing at $x=0$, $f(0)=-1$ is a local minimum value.

(b) (20 points) Find the intervals on which f is concave up/down and all inflection points of f.

Solution: In order to find where f is concave up/down, we find where the second derivative f'' is positive/negative. The method is the same as part (a), except now we work with the second derivative f''. Since f'' has a much simpler factored form than f', it has a much simpler sign table.

$$\begin{aligned}
f''(x) = 20(x+1)^3 &= 0\\
x+1 &= 0\\
x &= -1
\end{aligned}$$

	$(-\infty,-1)$	$(-1,\infty)$
$20(x+1)^3$	neg	pos
$f''(x)$	neg	pos
$f(x)$	C.D.	C.U.

Thus f is concave down on $(-\infty,-1)$ and concave up on $(-1,\infty)$. Since f changes from cancave down to cancave up at $x=-1$, the graph of f has an inflection point at $(-1, f(-1)) = (-1,3)$.

3. (20 points) A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1+N^2}$$

where k is a positive constant. What nitrogen level gives the largest yield?

Solution: Note that while Y is mathematically defined for all values of N, in this context where N represents the level of nitrogen in the soil, the domain should be restricted to $N \geq 0$. So we must find the absolute maximum value of Y on the interval $[0, \infty)$. Since this interval is not closed, the Extreme Value Theorem does not guarentee that there is an absolute maximum value for Y. Still, by finding where Y is increasing/decreasing, we hope to find an absolute maximum value. We proceed just as in question 1.

$$\begin{aligned} Y' = \frac{k(1+N^2) - kN(2N)}{(1+N^2)^2} &= 0 \text{ (quotient rule)} \\ \frac{k(1-N^2)}{(1+N^2)^2} &= 0 \\ \frac{k(1+N)(1-N)}{(1+N^2)^2} &= 0 \\ N &= \pm 1 \end{aligned}$$

Note that only the critical number $N = 1$ is in the domain $(0, \infty)$.

	$(0,1)$	$(1,\infty)$
k	pos	pos
$(1+N)$	pos	pos
$(1-N)$	pos	neg
$(1+N^2)^2$	pos	pos
Y'	pos	neg
Y	incr	decr

So Y is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. Thus Y has a local maximum at $N = 1$ with local maximum value $Y = 1/2$. Since Y is increasing on *all* of its domain to the left of $N = -1$ and decreasing on *all* of its domain to the right of $N = 1$, we see that Y also attains its *absolute* maximum at $N = 1$ with *absolute* maximum value $Y = 1/2$.

4. (20 points (bonus)) Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

Solution: Recall that the distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

So the distance from any point (x, y) and the point $(3, 0)$ is

$$\sqrt{(x-3)^2 + y^2}. \tag{1}$$

Now suppose the point (x, y) is on the curve $y = \sqrt{x}$. Then, replacing y with $\sqrt{x}$ in equation (1) gives

$$\sqrt{(x-3)^2 + \sqrt{x}^2}.$$

Let us set

$$f(x) = \sqrt{(x-3)^2 + \sqrt{x}^2} = \sqrt{x^2 - 5x + 9}.$$

Thus $f(x)$ is the distance between $(3, 0)$ and a point of the curve $y = \sqrt{x}$ as a function of x. Our job is to find the absolute minimum value of f on its domain $[0, \infty)$. Since this interval is not closed, the Extreme Value Theorem does not guarentee that there is an absolute minimum value for f. Still, by finding where f is increasing/decreasing, we hope to find an absolute minimum value. We proceed just as in question 1.

$$\begin{aligned} f'(x) = \frac{1}{2}(x^2 - 5x + 9)^{-1/2}(2x - 5) &= 0 \\ \frac{2x-5}{2\sqrt{x^2 - 5x + 9}} &= 0 \\ 2x - 5 &= 0 \\ x &= 5/2 \end{aligned}$$

	$(0, 5/2)$	$(5/2, \infty)$
$2x - 5$	neg	pos
$2\sqrt{x^2 - 5x + 9}$	pos	pos
$f'(x)$	neg	pos
$f(x)$	decr	incr

So f is decreasing on $(0, 5/2)$ and increasing on $(5/2, \infty)$. Thus f has a local minimum at $x = 5/2$. Since $f(x)$ is decreasing on *all* of its domain to the left of $x = 5/2$ and increasing on *all* of its domain to the right of $x = 5/2$, we see that $f(x)$ also attains its *absolute* minimum at $x = 5/2$. Therefore, the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$ has x-coordinate $5/2$, that is $(5/2, \sqrt{5/2})$.