CH. 10 MATHEMATICAL INDUCTION

(POWERFUL PROOF TECHNIQUE)

WHEN? TO PROVE OPEN STATEMENT $P(n)$ IS TRUE FOR $n = 1, 2, 3, \ldots$

i.e. $\forall n \in \mathbb{N}, P(n)$

i.e. $P(1) \wedge P(2) \wedge P(3) \wedge \ldots$

HOW? TWO STEPS: (1) **BASIS STEP**

PROVE $P(1)$ EXPLICITLY

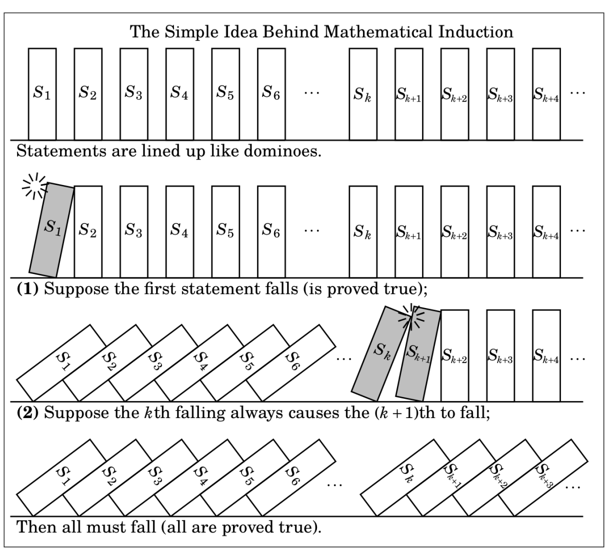
(2) **INDUCTIVE STEP**

GIVEN ANY $n \in \mathbb{N}$, PROVE THAT $P(n) \Rightarrow P(n+1)$

"WHENEVER $P(n)$ IS TRUE, $P(n+1)$ IS ALSO TRUE"

GENERALLY, THIS IS DONE DIRECTLY BY ASSUMING $P(n)$ IS TRUE (INDUCTIVE HYPOTHESIS) & THEN SHOWING THAT $P(n+1)$ MUST ALSO BE TRUE.

WHY DO THESE 2 STEPS PROVE THAT $P(n)$ IS TRUE FOR ALL $n \in \mathbb{N}$?



e.g. THE SUM OF THE FIRST n NATURAL NUMBERS IS $\dfrac{n(n+1)}{2}$.

i.e. $\boxed{1+2+3+\dots+n = \dfrac{n(n+1)}{2}}$ FOR ALL $n \in \mathbb{N}$.

This is $P(n)$

i.e. $1 = \dfrac{1(1+1)}{2}$, $1+2 = \dfrac{2(2+1)}{2}$,

$1+2+3 = \dfrac{3(3+1)}{2}$, $1+2+3+4 = \dfrac{4(4+1)}{2}$

PROOF: WE PROCEED BY INDUCTION.

(BASIS) FIRST WE SHOW STATEMENT IS TRUE FOR $n=1$.
OBSERVE THAT $\boxed{1 = \dfrac{1(1+1)}{2}.}$ $P(1)$ ✓

(INDUCTION) NOW WE SHOW IF THE STATEMENT IS TRUE FOR SOME $n \in \mathbb{N}$,
THEN IT IS TRUE FOR $n+1$.
WE DO SO DIRECTLY.
ASSUME $\boxed{1+2+3+\dots+n = \dfrac{n(n+1)}{2}.}$ $P(n)$

INDUCTIVE HYPOTHESIS

NOW WE MUST SHOW THAT

$$\boxed{1+2+3+\dots+n+(n+1) = \frac{(n+1)((n+1)+1)}{2}.}\quad P(n+1)$$

WE HAVE

$$\begin{aligned}
1+2+3+\dots+n+(n+1) &= \left(1+2+3+\dots+n\right)+(n+1)\\
&= \frac{n(n+1)}{2}+(n+1) \qquad \text{(BY INDUCTIVE HYP.)}\\
&= \frac{n(n+1)+2(n+1)}{2}\\
&= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}
\end{aligned}$$

IT FOLLOWS BY INDUCTION THAT $1+2+3+\dots+n = \dfrac{n(n+1)}{2}$ FOR ALL $n \in \mathbb{N}$. ∎

Note: More generally, induction can be used to show that some statement $P(n)$ is true for all integers $\geq m$.
(m does not have to be 1)

$$P(m) \wedge P(m+1) \wedge P(m+2) \wedge \dots$$

Now $P(m)$ is the base case.
Everything else is the same.

Alt: Induction step $P(n) \Rightarrow P(n+1)$ is equivalent to $P(n-1) \Rightarrow P(n)$.
Style choice.

If true for some integer then true for next integer.

ex. For all $n \in \mathbb{N}$, $3^n < 3^{n+1} - 3^{n-1} - 2$.

Proof. We proceed by induction.
First, observe that the statement is true for $n=1$:

Step 1 (Basis)

$$3 < 3^2 - 3^0 - 2 = 6.$$

Step 2 (Induction)

Now we show that if $3^n < 3^{n+1} - 3^{n-1} - 2$ for some $n \in \mathbb{N}$, then $3^{n+1} < 3^{(n+1)+1} - 3^{(n+1)-1} - 2$

Assume $3^n < 3^{n+1} - 3^{n-1} - 2$. (Inductive hypothesis)

Then
$$\begin{aligned} 3^{n+1} &= 3 \cdot 3^n \\ &< 3(3^{n+1} - 3^{n-1} - 2) \\ &= 3^{n+2} - 3^n - 6 \\ &= 3^{(n+1)+1} - 3^{(n+1)-1} - 6 \\ &< 3^{(n+1)+1} - 3^{(n+1)-1} - 2. \ \blacksquare \end{aligned}$$

Note that we could use $\leq$ for every line.

Thus, if $3^n < 3^{n+1} - 3^{n-1} - 2$ then $3^{n+1} < 3^{(n+1)+1} - 3^{(n+1)-1} - 2$. $\blacksquare$

ex.

18. Suppose $A_1, A_2, \ldots A_n$ are sets in some universal set U, and $n \geq 2$. Prove that $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.

PROOF: WE PROCEDE BY INDUCTION.
FIRST, OBSERVE THAT WHEN $n=2$, THE STATEMENT IS TRUE.
IN FACT, FOR $n=2$, THE STATEMENT IS DE MORGAN'S LAW.
(FOR A PROOF SEE NOTES TO CH. 6)

NOW LET US ASSUME $\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$
AND TRY TO SHOW THAT

$$\overline{A_1 \cup A_2 \cup \ldots \cup A_n \cup A_{n+1}} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n} \cap \overline{A_{n+1}}.$$

WE HAVE

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \ldots \cup A_n \cup A_{n+1}} &= \overline{(A_1 \cup A_2 \cup \ldots \cup A_n) \cup A_{n+1}} \\
&= \overline{(A_1 \cup A_2 \cup \ldots \cup A_n)} \cap \overline{A_{n+1}}
\end{aligned}$$

(CASE $n=2$)

$$= \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n} \cap \overline{A_{n+1}}. \quad \blacksquare$$

(BY INDUCTIVE HYPOTHESIS)