CH 11 : RELATIONS

§ 11.1 RELATIONS

FAMILIAR WAYS MATH OBJECTS CAN BE "RELATED" :

$3 < 4$ $\frac{21}{14} = \frac{3}{2}$ $8 \mid 24$ $7 \nmid 24$

$8 \equiv 13 \pmod{5}$ $2 \in \mathbb{N}$ $\mathbb{N} \subseteq \mathbb{Z}$ $A \neq \emptyset$

$\boxed{(\text{OBJECT } 1) \quad \text{RELATION} \quad (\text{OBJECT } 2)}$

SYMBOL EXPRESSING SOME RELATIONSHIP BETWEEN OBJECTS

$<, \leq, >, \geq, =, \neq$
$\in, \subseteq, \notin, \nsubseteq$, ETC.

OTHER RELATIONS ?

NUMBERS

- SAME / DIFFERENT PARITY
- MULTIPLICATIVE INVERSE
- HAVE SAME ABS. VAL.
- HAVE SAME SIGN
- ONLY COMMON FACTOR IS ONE

∞ MANY

SETS

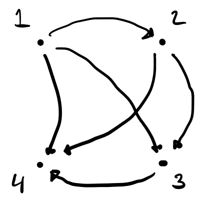
- HAVE SAME NUMBER OF ELEMENTS
- HAVE EMPTY INTERSECTION
- HAVE NON-EMPTY INTERSECTION
- ARE COMPLEMENTS

GOAL: DEFINING WHAT A RELATION IS IN A WAY THAT INCLUDES ALL OF THE ABOVE AS JUST SPECIFIC EXAMPLES.

THEN USE THIS DEFINITION TO BEGIN STUDYING ALL RELATIONS AT ONCE, IN GENERAL.

ex. LET $A = \{1, 2, 3, 4\}$
WRITE DOWN ALL OF THE "$<$"-RELATION STATEMENTS BETWEEN ELEMENTS IN A.

$$\boxed{\begin{array}{l} 1<2,\ 1<3,\ 1<4, \\ 2<3,\ 2<4 \\ 3<4 \end{array}}$$



ARROW FROM a TO b IFF $a<b$.

SO THE RELATION "$<$" ON A CAN BE ENCODED AS A SET

$$R_< = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\} \subseteq A \times A$$

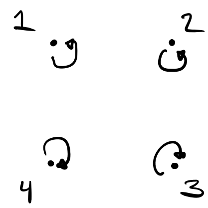
WHERE $a<b \iff (a,b) \in R_<$

ex. WHAT SUBSET OF $A \times A$ ENCODES THE RELATION "$=$" ON SET A

$1=1$, $2=2$, $3=3$, $4=4$

$$\boxed{R_= = \{(1,1), (2,2), (3,3), (4,4)\}} \subseteq A \times A$$

$a=b \iff (a,b) \in R_=$

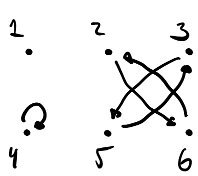


Definition 11.1 A **relation** on a set A is a subset $R \subseteq A \times A$. We often abbreviate the statement $(x,y) \in R$ as xRy. The statement $(x,y) \notin R$ is abbreviated as $x \not R\, y$.

ex. LET $A = \{1,2,3,4,5,6\}$ &

$$R = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$$

CAN YOU DESCRIBE THE RELATION IN WORDS? ADD TO 8
USING SET-BUILDER NOTATION?
$R = \{(a,b) \in A \times A : a+b=8\}$



ex. USE SET-BUILDER NOTATION TO DESCRIBE THE RELATION $<$ ON $\mathbb{Z}$.

$$R_{<} = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} : b - a \in \mathbb{N} \}$$

ex. USE SET-BUILDER NOTATION TO DESCRIBE THE RELATION $\leq$ ON $\mathbb{R}$.

$$R_{<} = \{ (a,b) \in \mathbb{R} \times \mathbb{R} : |b-a| = b-a \}$$

$$\left(\text{OR} \quad \sqrt{b-a} \in \mathbb{R} \right)$$

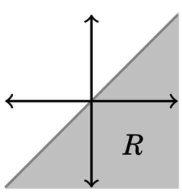
ex. IF $|A| = 5$, HOW MANY DIFFERENT RELATIONS ARE THERE ON THE SET A?

$\rightarrow$ HOW MANY DIFFERENT SUBSETS OF $A \times A$ ARE THERE?

$$|A \times A| = |A| \times |A| = 5 \times 5 = 25$$

$$|\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^{25}$$

In the following exercises, subsets R of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ or $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ are indicated by gray shading. In each case, R is a familiar relation on $\mathbb{R}$ or $\mathbb{Z}$. State it.

12.


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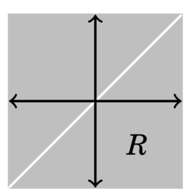

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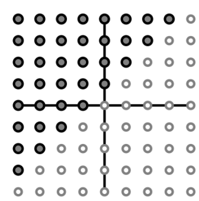

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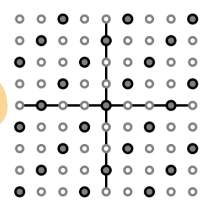


ex. RECALL: ON $A = \{1,2,3,4\}$

$$a < b \iff (a,b) \in R_{<} = \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \}$$

$$a = b \iff (a,b) \in R_{=} = \{ (1,1), (2,2), (3,3), (4,4) \}$$

WHAT RELATION DOES $R_{<} \cup R_{=}$ DEFINE? $\leq$

NOTE: RELATIONS ARE SETS AND SO WE CAN APPLY SET OPERATORS $\cap, \cup, -$ TO RELATIONS.

§11.2 Properties of Relations

Relations produce statements: $8 < 11$ (True)
$81 < 73$ (False)

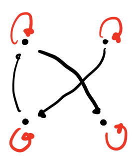
& open sentences: $x < y$ (truth value depends on x, y)

So we may combine relational expressions with logical operators ($\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\sim$, etc.) & quantifiers ($\forall$, $\exists$).

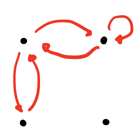
Definition 11.2 Suppose R is a relation on a set A.

1. Relation R is **reflexive** if xRx for every $x \in A$.
   That is, R is reflexive if $\forall x \in A, xRx$.
2. Relation R is **symmetric** if xRy implies yRx for all $x, y \in A$.
   That is, R is symmetric if $\forall x, y \in A, xRy \Rightarrow yRx$.
3. Relation R is **transitive** if whenever xRy and yRz, then also xRz.
   That is, R is transitive if $\forall x, y, z \in A, ((xRy) \wedge (yRz)) \Rightarrow xRz$.

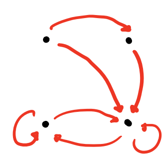
Reflexive:



Symmetric:



Transitive



These loops are necessary! See def of transitive with $z = x$.

ex.

Relations on $\mathbb{Z}$:	$<$	$\leq$	$=$	$\mid$	$\nmid$	$\neq$
Reflexive	no	yes	yes	yes	no	no
Symmetric	no	no	yes	no	no	yes
Transitive	yes	yes	yes	yes	no	no

Theorem 1. *The relation $|$ (divides) on the set $\mathbb{Z}$ is reflexive and transitive but not symmetric.*

Proof. We have three statements to prove, and we will do so one at a time.

First, to prove that the relation $|$ on $\mathbb{Z}$ is reflexive, we must show that for all $a \in \mathbb{Z}$, $a|a$.

$$\forall a \in \mathbb{Z},\ a|a.$$

So let $a \in \mathbb{Z}$. Since $a = a \cdot 1$ and $1 \in \mathbb{Z}$, this shows that $a|a$ by defintition.

Second, to show that the relation $|$ on $\mathbb{Z}$ is transitive, we must show that

$$\forall a, b, c \in \mathbb{Z}, \text{ if } a|b \text{ and } b|c, \text{ then } a|c.$$

Suppose $a, b, c \in \mathbb{Z}$, $a|b$, and $b|c$. Then, by definition, there exist $m, n \in \mathbb{Z}$ such that $b = ma$ and $c = nb$. It follows that $c = n(ma) = (nm)a$. Since $mn \in \mathbb{Z}$, this shows that $a|c$.

Finally, to show that $|$ is not symmetric on $\mathbb{Z}$ we must show

$$\sim (\forall a, b \in \mathbb{Z}, \text{ if } a|b \text{ then } b|a).$$

That is, we must show

$$\exists a, b \in \mathbb{Z}, \text{ such that } a|b \text{ and } b \nmid a.$$

It is enough to provide an example. Let $a = 3$ and $b = 6$. Since $6 = 2 \cdot 3$, and $2 \in \mathbb{Z}$, we see that $3|6$. Now assume for sake of contradiction that $6|3$. Then $3 = 6n$ for some $n \in \mathbb{Z}$. However, this implies that $3/6 = n$ is an integer. Since $3/6 = 1/2$ is *not* an integer, this is a contradiction. Thus, our assumption that $3|6$ must be false. Therefore, since 3 and 6 are integers such that $3|6$ and $6 \nmid 3$, we see that the relation $|$ on $\mathbb{Z}$ is *not* symmetric. $\square$

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