§11.3 Equivalence Relations

Def: A relation R on a set A is an equivalence relation if it is
(1) reflexive
(2) symmetric
(3) transitive

$R \subseteq A \times A$

$\forall x \in A, \; xRx$
$\forall x, y \in A, \; xRy \Rightarrow yRx$
$\forall x, y, z \in A, \; (xRy \wedge yRz) \Rightarrow xRz$

Some sort of "sameness"

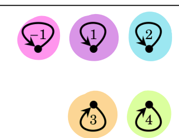
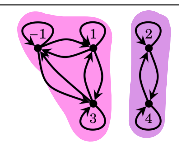
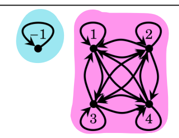
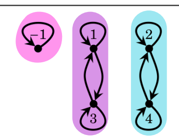
Relation R	Diagram	Equivalence classes (see next page)
"is equal to" (=) $R_1 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4)\}$		$\{-1\}, \{1\}, \{2\},$ $\{3\}, \{4\}$
"has same parity as" $R_2 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),$ $(-1,1),(1,-1),(-1,3),(3,-1),$ $(1,3),(3,1),(2,4),(4,2)\}$		$\{-1,1,3\}, \{2,4\}$
"has same sign as" $R_3 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),$ $(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),$ $(3,4),(4,3),(2,3),(3,2),(2,4),(4,2)\}$		$\{-1\}, \{1,2,3,4\}$ $[1]=[2]=[3]=[4]$
"has same parity and sign as" $R_4 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),$ $(1,3),(3,1),(2,4),(4,2)\}$		$\{-1\}, \{1,3\}, \{2,4\}$ $[-1]$ $[1]=[3]$ $[2]=[4]$

Figure 11.2. Examples of equivalence relations on the set $A = \{-1, 1, 2, 3, 4\}$

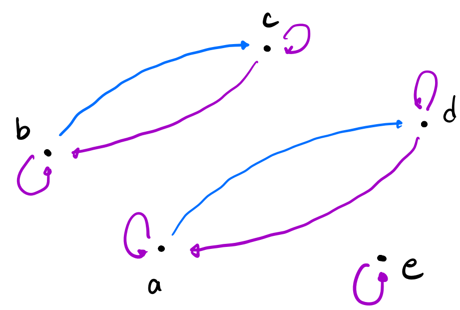
Observation: An equivalence relation on A divides A into (disjoint) subsets (called equivalence classes)

Def Suppose R is an equivalence relation on a set A. Given any element $a \in A$, the equivalence class containing a is the subset $\{x \in A : xRa\}$ of A consisting of all elements of A that relate to a. This set is denoted $[a]$.

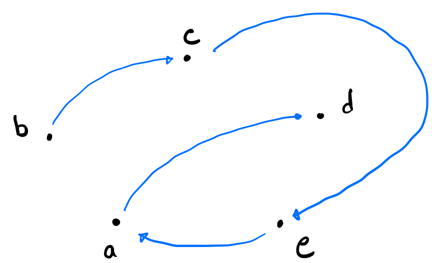
$$[a] = \{x \in A : xRa\}$$

ex. Let R be equiv. relation $\equiv \pmod 5$ on $\mathbb{Z}$.
Describe the equivalence classes.

3. Let $A = \{a, b, c, d, e\}$. Suppose R is an equivalence relation on A. Suppose R has three equivalence classes. Also aRd and bRc. Write out R as a set.



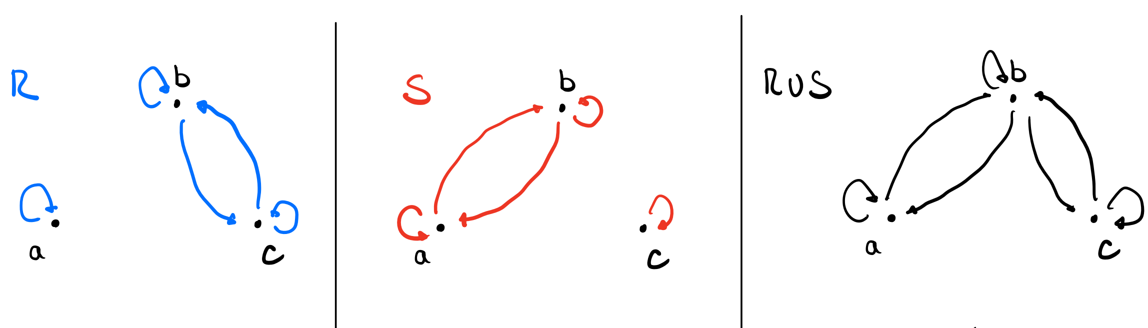
4. Let $A = \{a, b, c, d, e\}$. Suppose R is an equivalence relation on A. Suppose also that aRd and bRc, eRa and cRe. How many equivalence classes does R have?



... Just 1

12. Prove or disprove: If R and S are two equivalence relations on a set A, then $R \cup S$ is also an equivalence relation on A.

Let $A = \{a, b, c\}$



Reflexive ✓

Symmetric ✓

Not transitive: $\underbrace{\big((a,b) \in R \cup S \wedge (b,c) \in R \cup S\big)}_{\text{True}} \Rightarrow \underbrace{(a,c) \in R \cup S}_{\text{False}}$ is false

§11.4 Equivalence Classes and Partitions

Thm (11.1) Suppose R is an equivalence relation on a set A and $a, b \in A$.

$$[a] = [b] \iff aRb.$$

Proof: ($\Rightarrow$) Suppose $[a] = [b]$.
Since R is reflexive, aRa and so
$a \in \{x \in A : xRa\} = [a] = [b] = \{x \in A : xRb\}$.
$\therefore aRb$.

($\Leftarrow$) We now show (1) $aRb \Rightarrow [a] \subseteq [b]$, and
(2) $aRb \Rightarrow [b] \subseteq [a]$.

(1) Suppose aRb and $c \in [a]$. (We show $c \in [b]$.)

Since $c \in [a] = \{x \in A : xRa\}$, cRa.
Now cRa and $aRb \Rightarrow cRb$ by transitive prop. of R.
$\therefore c \in \{x \in A : xRb\} = [b]$.

(2) Suppose aRb and $c \in [b]$. (We show $c \in [a]$.)

Since $c \in [b] = \{x \in A : xRb\}$, cRb.
Since aRb and R is symmetric, bRa.
Now cRb and $bRa \Rightarrow cRa$ by transitive prop. of R.
$\therefore c \in \{x \in A : xRa\} = [a]$. ∎

Def: A **partition** of a set A is a set of non-empty subsets of A such that

(1) The union of all the subsets equals A, and
(2) The intersection of any 2 different subsets is $\emptyset$.

ex. List all possible partitions of $\{a, b, c\}$.

1. $\{\{a\}, \{b\}, \{c\}\}$
2. $\{\{a, b\}, \{c\}\}$
3. $\{\{a, c\}, \{b\}\}$
4. $\{\{a\}, \{b, c\}\}$
5. $\{\{a, b, c\}\}$

Thm (11.2) Suppose R is an equivalence relation on a set A. The set $\{[a] : a \in A\}$ of equivalence classes of R forms a partition of A.

Proof: We must show (1) $\bigcup_{a \in A} [a] = A$, and

(2) if $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$.

(1) (a) $\bigcup_{a \in A} [a] \subseteq A$: Let $c \in \bigcup_{a \in A} [a]$.
Then $\exists a \in A$ s.t. $c \in [a]$.
Since $[a] \subseteq A$, $c \in A$.

(b) $A \subseteq \bigcup_{a \in A} [a]$: Let $c \in A$.
Since R is an equivalence relation, R is reflexive.
$\therefore$ cRc and so $c \in \{x \in A : xRc\} = [c]$.
$\therefore$ $c \in \bigcup_{a \in A} [a]$ (namely $a = c$).

(2) (contrapositive) Suppose $[a] \cap [b] \neq \emptyset$. (We show $[a] = [b]$.)

Then $\exists x \in A$ such that $x \in [a]$ and $x \in [b]$.
We have xRa and xRb.
Since R is symmetric, aRx.
Thus aRx and xRb, so aRb (R is transitive).
$\therefore$ Thm 11.1 implies $[a] = [b]$.

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