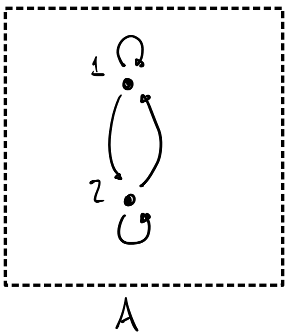
Ch 12: Functions

§12.1 Functions

(recall, from §11.6) **Def:** A relation from a set A to a set B is a subset $R \subseteq A \times B$.

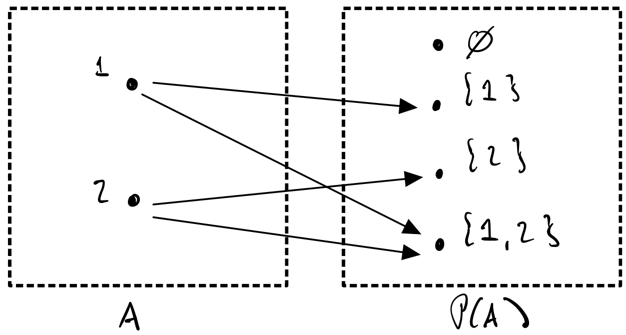
Note the difference:

Relation on A: $R \subseteq A \times A$



"has the same sign"

Relation from A to B: $R \subseteq A \times B$



" is an element of "

Q: Can a relation from A to B be an equivalence relation?

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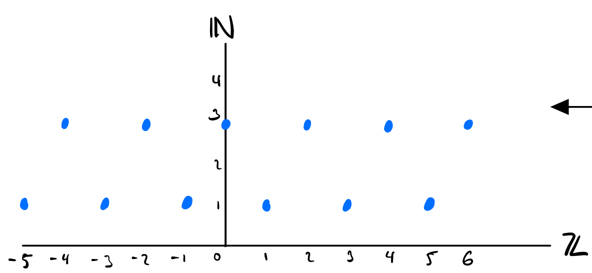
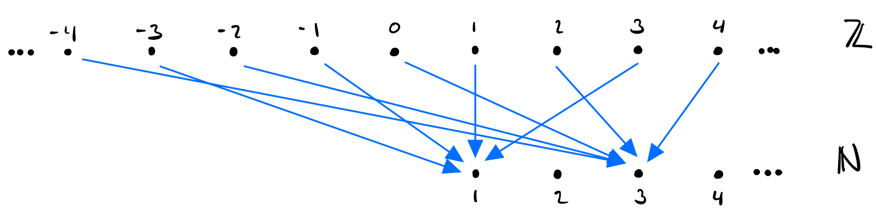
ex. Consider function $f(n) = 2 + (-1)^n$ that turns integers into natural numbers.



← its graph is a relation from $\mathbb{Z}$ to $\mathbb{N}$

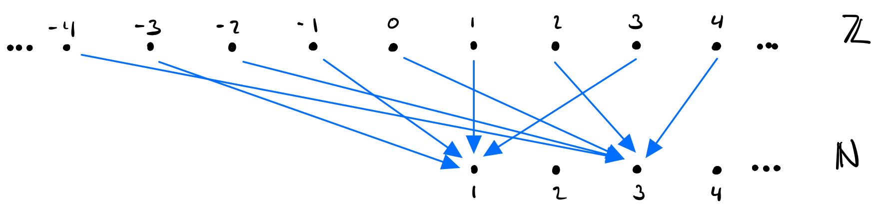
$f = \{ (n, 2 + (-1)^n) : n \in \mathbb{Z} \} \subseteq \mathbb{Z} \times \mathbb{N}$

$= \{ \ldots, (-1,1), (0,3), (1,1), (2,3), \ldots \}$



Def: Suppose A & B are sets.
A function f from A to B $(f : A \to B)$ is a relation from A to B, $f \subseteq A \times B$, such that for each $a \in A$ the relation f contains exactly one ordered pair of the form (a, b).
The statement $(a, b) \in f$ is abbreviated $f(a) = b$.

The set A is called the **Domain** of f.
The set B is called the **Codomain** of f.
The **Range** of f is the set $\{f(a) : a \in A\} = \{b \in B : (a,b) \in f\}$



DOMAIN

CODOMAIN ("TARGET")

$\{1, 3\}$ RANGE ("HITS")

Note: The range is a subset of the codomain, and codomain is not unique – it could be <u>any</u> set that contains the range.

ex. Let $S = \{x : x$ is a student in this class$\}$
$M = \{$Jan, Feb, ..., Dec$\}$

Define $f : S \to M$ as $f(x) =$ Birthmonth of x, that is
$f = \{(x, \text{Birthmonth of } x) : x \in S\} \subseteq S \times M$

What is the Domain, Codomain, & Range?

7. Consider the set $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 3x + y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.

$f = \{(x, 4 - 3x) : x \in \mathbb{Z}\}$ ✓

ex. What about $f = \{(x,y) \in \mathbb{Z}^2 : \underbrace{3x + y^2 = 4}_{y = \pm\sqrt{4 - 3x}}\}$?

✗

TYPICALLY WRITTEN $\varphi(m,n)$

ex. LET $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ BE DEFINED AS $\varphi((m,n)) = 12m + 4n$.
THAT IS, $\varphi = \{ ((m,n), 12m + 4n) : (m,n) \in \mathbb{Z}^2 \}$.
WHAT ARE DOMAIN, CODOMAIN, AND RANGE?
$\mathbb{Z}^2$ $\quad$ $\mathbb{Z}$

$$\text{RANGE} = \{ 12m + 4n : (m,n) \in \mathbb{Z}^2 \} = \{ 4(3m+n) : (m,n) \in \mathbb{Z}^2 \}$$

$$\subseteq \{ 4p : p \in \mathbb{Z} \}$$

$$\text{AND } \{ 4p : p \in \mathbb{Z} \} = \{ 4(3p - 2p) : p \in \mathbb{Z} \} = \{ 12p + 4(-2p) : p \in \mathbb{Z} \}$$

$$\subseteq \{ 12m + 4n : (m,n) \in \mathbb{Z}^2 \}$$

$$\therefore \ \text{RANGE} = \{ 4p : p \in \mathbb{Z} \}$$

NOTE: WHEN ARE TWO FUNCTIONS $f : A \to B$ & $g : C \to D$ EQUAL?

FUNCTIONS ARE SETS! $f \subseteq A \times B$ & $g \subseteq C \times D$

e.g. $f = \{ (a,1), (b,4), (c,4) \}$
$g = \{ (c,4), (a,1), (b,4) \}$

THESE FUNCTIONS ARE EQUAL AS SETS.
NOTE THAT THE SET OF ALL FIRST COMPONENTS OF ORDERED PAIRS (DOMAINS) MUST BE EQUAL, AND $f(x) = g(x)$ FOR ALL x IN DOMAIN.

Definition 12.3 Two functions $f : A \to B$ and $g : A \to D$ are **equal** if $f = g$ (as sets). Equivalently, $f = g$ if and only if $f(x) = g(x)$ for every $x \in A$.

§12.2 Injective & Surjective Functions

Definition 12.4 A function $f : A \to B$ is:

1. **injective** (or *one-to-one*) if for all $a, a' \in A$, $a \neq a'$ implies $f(a) \neq f(a')$;
2. **surjective** (or *onto* B) if for every $b \in B$ there is an $a \in A$ with $f(a) = b$;
3. **bijective** if f is both injective and surjective.

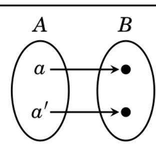
INJECTIVE: $\forall a, a' \in A,\ a \neq a' \Rightarrow f(a) \neq f(a')$.

NOT INJECTIVE: $\exists a, a' \in A,\ a \neq a' \wedge f(a) = f(a')$.

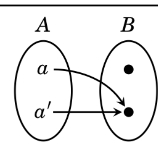
SURJECTIVE: $\forall b \in B, \exists a \in A,\ f(a) = f(b)$.

NOT SURJECTIVE: $\exists b \in B, \forall a \in A,\ f(a) \neq f(b)$.

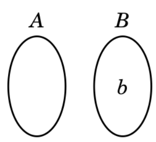
Injective means that for any two $a, a' \in A$, this happens...



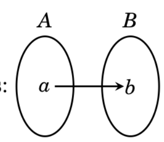
...and not this:



Surjective means that for any $b \in B$...



...this happens:



1. Let $A = \{1,2,3,4\}$ and $B = \{a,b,c\}$. Give an example of a function $f : A \to B$ that is neither injective nor surjective.

How to show a function $f : A \to B$ is injective:

Direct approach:	**Contrapositive approach:**
Suppose $a, a' \in A$ and $a \neq a'$.	Suppose $a, a' \in A$ and $f(a) = f(a')$.
$\vdots$	$\vdots$
Therefore $f(a) \neq f(a')$.	Therefore $a = a'$.

How to show a function $f : A \to B$ is surjective:

Suppose $b \in B$.
[Prove there exists $a \in A$ for which $f(a) = b$.]

5. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Verify whether this function is injective and whether it is surjective.

INJECTIVE ✓
SURJECTIVE ⊗

13. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x,y) = (xy, x^3)$. Is f injective? Is it surjective? Bijective? Explain.

NOT INJECTIVE: $(0,1) \neq (0,2)$ AND $f(0,1) = f(0,2) = (0,0)$

NOT SURJECTIVE: ASSUME FOR SAKE OF CONTRADICTION THAT $f(x,y) = (1,0)$.

THEN $xy = 1$ AND $x^3 = 0$

BUT $x^3 = 0 \Rightarrow x = 0 \Rightarrow xy = 0 \Rightarrow \Leftarrow$

$\therefore \nexists (x,y) \in \mathbb{R}^2, f(x,y) = (1,0)$.

14. Consider the function $\theta : \mathscr{P}(\mathbb{Z}) \to \mathscr{P}(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$. Is θ injective? Is it surjective? Bijective? Explain.

BIJECTIVE ✓