§12.3 The Pigeonhole Principle Revisited

At night a flock of pigeons all sleep in a collection of pigeonholes.

Define a function $f: \text{Pigeons} \to \text{Pigeonholes}$ as $f(x) =$ the pigeonhole that x sleeps in.

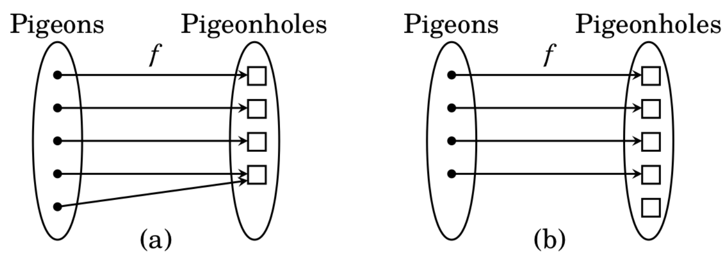


Figure 12.4. The pigeonhole principle

If # pigeons > # pigeonholes then at least one pigeonhole will contain at least 2 pigeons.

If # pigeons < # pigeonholes then at least one pigeonhole will be empty.

> **The Pigeonhole Principle** (function version)
> Suppose A and B are finite sets and $f: A \to B$ is any function.
> 1. If $|A| > |B|$, then f is not injective.
> 2. If $|A| < |B|$, then f is not surjective.

ex. There exist 2 people with the exact same number of freckles.

Proof: Define $f: \text{People} \to \{0, 1, \dots, 1{,}000{,}000{,}000\}$

$f(x) = \#$ freckles on person x

I assume no one has more than 1,000,000,000 freckles.

Since $\underbrace{|\text{People}| > 1{,}000{,}000{,}001}_{\text{I assume this is true.}}$, f is not injective.

Thus $\exists\, x, y \in \text{People}$ s.t. $f(x) = f(y)$.

ex. IF A IS ANY SET OF 6 INTEGERS BETWEEN 1 & 10, THEN THERE EXIST 2 DIFFERENT SUBSETS $X \subseteq A$ & $Y \subseteq X$ FOR WHICH THE SUM OF ELEMENTS IN X EQUALS THE SUM OF ELEMENTS IN Y.

e.g. IF $A = \{6, 8, 1, 3, 9, 2\}$

$X = \{9\}$, $Y = \{6, 3\}$ OR

$X = \{8, 3\}$, $Y = \{9, 2\}$

PROOF: LET $A \subseteq \{1, 2, \ldots, 10\}$ WITH $|A| = 6$.

NOTICE THAT IF WE ADD UP ALL THE ELEMENTS OF A, THE SUM IS LESS THAN $5 + 6 + 7 + 8 + 9 + 10 = 45$.

DEFINE $f : \mathcal{P}(A) \to \{0, 1, 2, \ldots, 45\}$ AS

$$f(X) = \sum_{x \in X} x$$ = THE SUM OF ALL ELEMENTS IN X, WHERE $X \subseteq A$.

THAT IS, $f = \{(X, n) \in \mathcal{P}(A) \times \{0, 1, \ldots, 45\} : \sum_{x \in X} x = n\}$.

THEN $|\mathcal{P}(A)| = 2^{|A|} = 2^6 = 64$, AND $|\{0, 1, \ldots, 45\}| = 46$.

$\therefore$ BY PIGEONHOLE PRINCIPLE, f CANNOT BE INJECTIVE.

THERE MUST EXIST TWO DISTINCT SUBSETS $X, Y \subseteq A$ (ELEMENTS OF $\mathcal{P}(A)$) SUCH THAT $f(X) = f(Y)$.

TRY FOR FUN!:

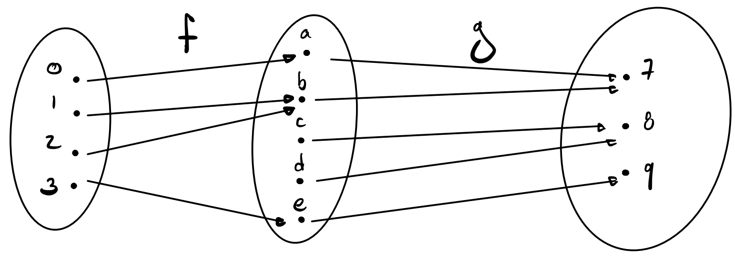
4. Consider a square whose side-length is one unit. Select any five points from inside this square. Prove that at least two of these points are within $\frac{\sqrt{2}}{2}$ units of each other.

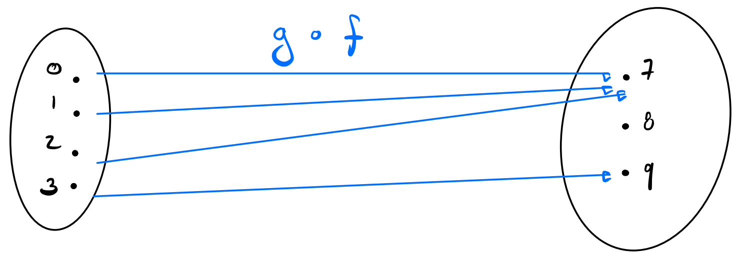
6. Given a sphere S, a *great circle* of S is the intersection of S with a plane through its center. Every great circle divides S into two parts. A *hemisphere* is the union of the great circle and one of these two parts. Prove that if five points are placed arbitrarily on S, then there is a hemisphere that contains four of them.

§12.4 Composition

Definition 12.5 Suppose $f: A \to B$ and $g: B \to C$ are functions with the property that the codomain of f equals the domain of g. The **composition** of f with g is another function, denoted as $g \circ f$ and defined as follows: If $x \in A$, then $g \circ f(x) = g(f(x))$. Therefore $g \circ f$ sends elements of A to elements of C, so $g \circ f: A \to C$.

e.g.





- RANGE OF $f \subseteq$ DOMAIN OF g (CODOMAIN OF f = DOMAIN OF g)
- COMPOSITIONS ARE CARRIED OUT RIGHT-TO-LEFT :

$$g \circ f = \underbrace{g(\underbrace{f(x)}_{1^{st}})}_{2^{nd}}$$

ex. SUPPOSE $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7\}$, $C = \{8, 9\}$.

$f: A \to B$, $f = \{(1,6), (2,5), (3,5), (4,7)\}$.

$g: B \to C$, $g = \{(5,8), (6,8), (7,9)\}$.

FIND $g \circ f$. WHY IS $f \circ g$ UNDEFINED ?

ex. Suppose $f: A \to B$ & $g: B \to C$ are injective, then $g \circ f$ is injective.

Proof: Let $a, a' \in A$.
We need to show that if $a \neq a'$, then $g \circ f(a) \neq g \circ f(a')$.

Let us prove the contrapositive statement:
If $g \circ f(a) = g \circ f(a')$ then $a = a'$.

Assume $g \circ f(a) = g \circ f(a')$. Since g is injective, $f(a) = f(a')$.

Since f is injective, $a = a'$. ∎

ex. Suppose $f: A \to B$ & $g: B \to C$ are surjective, then $g \circ f$ is surjective.

Proof: Let $c \in C$. We need to show $\exists\, a \in A$ s.t. $g \circ f(a) = c$.

Since g is surjective, $\exists\, b \in B$ s.t. $g(b) = c$.

Since f is surjective, $\exists\, a \in A$ s.t. $f(a) = b$.

Thus, $g \circ f(a) = g(f(a)) = g(b) = c$.

Corollary: If $f: A \to B$ & $g: B \to C$ are bijective.

Then $g \circ f$ is bijective.

THM: Let $f: A \to B$, $g: B \to C$, $h: C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

That is, function composition is <u>associative</u>

PROOF: Let $a \in A$ be any element in A.
We need to show that $h \circ (g \circ f)(a) = (h \circ g) \circ f(a)$.

Set $f(a) = b \in B$, $g(b) = c \in C$, and $h(c) = d \in D$.

Then $g \circ f(a) = g(f(a)) = g(b) = c$ and

$$h \circ g(b) = h(g(b)) = h(c) = d.$$

Now $h \circ (g \circ f)(a) = h(g \circ f(a)) = h(c) = d = h \circ g(b)$

$$= (h \circ g) \circ f(a).$$

∎