§12.5 Inverse Functions

Definition 12.6 For a set A, the **identity function** on A is the function $i_A : A \to A$ defined as $i_A(x) = x$ for every $x \in A$.

That is, $i_A = \{ (a,a) : a \in A \}$.

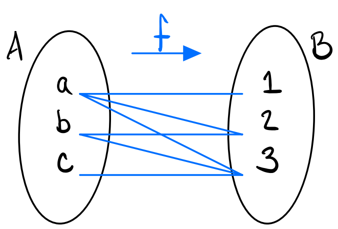
e.g.

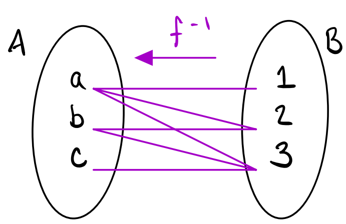
Reflexive ✓
Symmetric ✓
Transitive ✓

Note: i_A is a relation on A. It is the familiar relation "$=$".

Definition 12.7 Given a relation R from A to B, the **inverse relation of** R is the relation from B to A defined as $R^{-1} = \{(y,x) : (x,y) \in R\}$. In other words, the inverse of R is the relation R^{-1} obtained by interchanging the elements in every ordered pair in R.

ex.

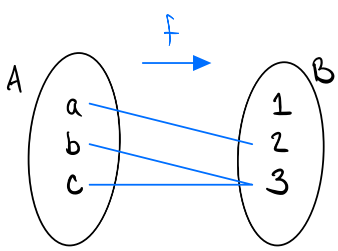


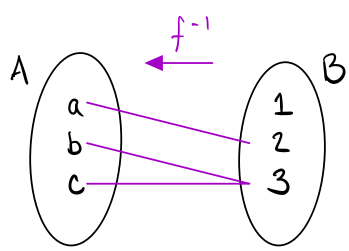


$f = \{ \underbrace{(a,1)(a,2)}_{\text{not a function}}, (a,3), (b,2), (b,3), (c,3) \}$

$f^{-1} = \{ (1,a), (2,a), (3,a), (2,b), \underbrace{(3,b), (3,c)}_{\text{not a function}} \}$

ex.





$f = \{ (a,2), (b,3), (c,3) \}$

$f : A \to B$ is a function

$f^{-1} = \{ (2,a), (3,b), (3,c) \}$

$f^{-1} : B \to A$ is not a function

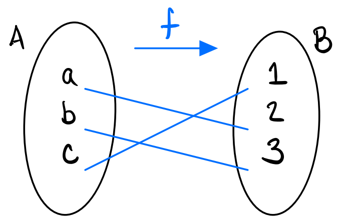
f is not injective ⟶ Two elements of f^{-1} have same 1^{st} component

f is not surjective ⟶ Not all elements of B appear as 1^{st} component

RECALL:

Definition 12.1 Suppose A and B are sets. A **function** f from A to B (denoted as $f: A \to B$) is a relation $f \subseteq A \times B$ from A to B, satisfying the property that for each $a \in A$ the relation f contains exactly one ordered pair of form (a,b). The statement $(a,b) \in f$ is abbreviated $f(a) = b$.

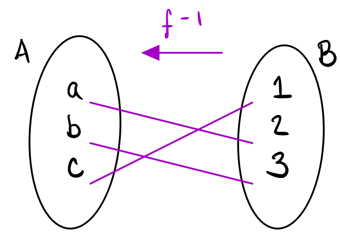
ex.



$f = \{(a,2),(b,3),(c,1)\}$

$f: A \to B$ IS A FUNCTION

f IS INJECTIVE & SURJECTIVE (BIJECTIVE)



$f^{-1} = \{(1,c),(2,a),(3,b)\}$

$f^{-1}: B \to A$ IS A FUNCTION

Theorem 12.3 Let $f: A \to B$ be a function. Then f is bijective if and only if the inverse relation f^{-1} is a function from B to A.

Definition 12.8 If $f: A \to B$ is bijective then its **inverse** is the function $f^{-1}: B \to A$. The functions f and f^{-1} obey the equations $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

$$\forall x \in A: (x, f(x)) \in f \Rightarrow (f(x), x) \in f^{-1} \Rightarrow f^{-1}(f(x)) = x$$

$$\forall x \in B: (x, f^{-1}(x)) \in f^{-1} \Rightarrow (f^{-1}(x), x) \in f \Rightarrow f(f^{-1}(x)) = x$$

4. The function $f: \mathbb{R} \to (0,\infty)$ defined as $f(x) = e^{x^3+1}$ is bijective. Find its inverse.

$y = \underbrace{e^{x^3+1}}$ SOLVE FOR x: $x = \underbrace{(\ln(y) - 1)^{1/3}}$

$y = f(x)$ $x = f^{-1}(y)$

$\therefore f^{-1}(x) = (\ln(x) - 1)^{1/3}$ (SWITCH THE VARIABLE IF YOU WANT)

7. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x,y) = ((x^2+1)y, x^3)$ is bijective. Then find its inverse.

INJECTIVE: SUPPOSE $f(x_1, y_1) = f(x_2, y_2)$

THEN $\left((x_1^2+1)y_1, x_1^3 \right) = \left((x_2^2+1)y_2, x_2^3 \right)$

$\Rightarrow \underbrace{(x_1^2+1)y_1 = (x_2^2+1)y_2}$ AND $\underbrace{x_1^3 = x_2^3}_{\Rightarrow\ x_1 = x_2}$

THEN $(x_1^2+1) = (x_2^2+1)$

$\Rightarrow \quad y_1 = y_2$

$\therefore (x_1, y_1) = (x_2, y_2)$.

SURJECTIVE: GIVEN $(a,b) \in \mathbb{R}^2$, SET $x = b^{1/3}$, $y = \dfrac{a}{b^{2/3}+1}$.

THEN $f(x,y) = f\left(b^{1/3}, \dfrac{a}{b^{2/3}+1}\right) = \left(\left((b^{1/3})^2+1\right)\dfrac{a}{b^{2/3}+1}, (b^{1/3})^3 \right)$

$= (a, b)$

INVERSE: LET $(a,b) = f(x,y)$: $(a,b) = \left((x^2+1)y, x^3 \right)$

WE HAVE $a = (x^2+1)y$ AND WE NOW SOLVE FOR x & y.
$b = x^3$

$x = b^{1/3}$, $y = \dfrac{a}{b^{2/3}+1}$

$\therefore (x,y) = f^{-1}(a,b) = \left(b^{1/3}, \dfrac{a}{b^{2/3}+1} \right)$

OR $f^{-1}(x,y) = \left(y^{1/3}, \dfrac{x}{y^{2/3}+1} \right)$

§12.6 Image & Preimage

Definition 12.9 Suppose $f : A \to B$ is a function.
1. If $X \subseteq A$, the **image** of X is the set $f(X) = \{f(x) : x \in X\} \subseteq B$.
2. If $Y \subseteq B$, the **preimage** of Y is the set $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.

Example 12.13 Let $f : \{s,t,u,v,w,x,y,z\} \to \{0,1,2,3,4,5,6,7,8,9\}$ be

$$f = \{(s,4),(t,8),(u,8),(v,1),(w,2),(x,4),(y,6),(z,4)\}.$$

This f is neither injective nor surjective, so it certainly is not invertible. Be sure you understand the following statements.

1. $f(\{s,t,u,z\}) = \{8,4\}$
2. $f(\{s,x,z\}) = \{4\}$
3. $f(\{s,v,w,y\}) = \{1,2,4,6\}$
4. $f(\emptyset) = \emptyset$
5. $f^{-1}(\{4\}) = \{s,x,z\}$
6. $f^{-1}(\{4,9\}) = \{s,x,z\}$
7. $f^{-1}(\{9\}) = \emptyset$
8. $f^{-1}(\{1,4,8\}) = \{s,t,u,v,x,z\}$

It is important to realize that the X and Y in Definition 12.9 are subsets (not elements!) of A and B. In Example 12.13 we had $f^{-1}(\{4\}) = \{s,x,z\}$, while $f^{-1}(4)$ is meaningless because the inverse function f^{-1} does not exist. And there is a subtle difference between $f(\{s\}) = \{4\}$ and $f(s) = 4$. Be careful.

Example 12.14 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$. Note that $f(\{0,1,2\}) = \{0,1,4\}$ and $f^{-1}(\{0,1,4\}) = \{-2,-1,0,1,2\}$. This shows that $f^{-1}(f(X)) \neq X$ in general.

ex.

10. Given $f : A \to B$ and subsets $Y, Z \subseteq B$, prove $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.

11. Given $f : A \to B$ and subsets $Y, Z \subseteq B$, prove $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.

(10) Proof:
$$\begin{aligned} f^{-1}(Y \cap Z) &= \{a \in A : f(a) \in Y \cap Z\} \\ &= \{a \in A : f(a) \in Y \wedge f(a) \in Z\} \\ &= \{a \in A : f(a) \in Y\} \cap \{a \in A : f(a) \in Z\} \\ &= f^{-1}(Y) \cap f^{-1}(Z). \end{aligned}$$

(11) Proof:
$$\begin{aligned} f^{-1}(Y \cup Z) &= \{a \in A : f(a) \in Y \cup Z\} \\ &= \{a \in A : f(a) \in Y \vee f(a) \in Z\} \\ &= \{a \in A : f(a) \in Y\} \cup \{a \in A : f(a) \in Z\} \\ &= f^{-1}(Y) \cup f^{-1}(Z). \end{aligned}$$