§2.7 Quantifiers

Suppose set $X = \{x_1, x_2, x_3, \dots\}$, i.e. $\{1, 3, 5, 7, \dots\}$

open sentence $P(x)$: x has this particular property, i.e. x is odd.

Statement: Every element of X has this property

$\hookrightarrow P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \dots = \forall x \in X,\ P(x)$

For all — For every — For each

Universal quantifier

Statement: At least one of the elements of X has the property

$\hookrightarrow P(x_1) \vee P(x_2) \vee P(x_3) \vee \dots = \exists x \in X,\ P(x)$

There exists a — There is a — For some

Existential quantifier

$\forall$ & $\exists$ are quantifiers – they specify the quantity of the variable that follows them: all or some

ex. "Every integer multiple of π is a solution to $\sin x = 0$."

$\forall n \in \mathbb{Z},\ \sin(n\pi) = 0.$

$\forall n \in \mathbb{Z},\ S(n)$
$S(n): \sin(n\pi) = 0$

ex. "There is a prime number greater than 100."

$\exists n \in \mathbb{N},\ (n \text{ is prime}) \wedge (n > 100)$

$\exists n \in \mathbb{Z},\ P(n) \wedge G(n)$
$P(n): n \text{ is prime}$
$G(n): n > 100$

Problem 3. Translate the following statements into symbolic logic. The universe of discourse is $\mathbb{R}$.

(i) The identity element for addition is 0.
(ii) Every real number has an additive inverse.
(iii) Negative numbers do not have square roots.
(iv) Every positive number has exactly two square roots.

Solution.

(i) $\forall x \in \mathbb{R},\ x + 0 = x$.
(ii) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y + x = 0$.
(iii) $\forall x \in \mathbb{R},\ x < 0 \Rightarrow\ \sim (\exists y \in \mathbb{R},\ y^2 = x)$.
(iv) $\forall x \in \mathbb{R},\ x > 0 \Rightarrow (\exists y_1, y_2, \in \mathbb{R},\ y_1^2 = y_2^2 = x \wedge y_1 \neq y_2) \wedge\ \sim (\exists z \in \mathbb{R},\ z \neq y_1 \wedge z \neq y_2 \wedge z^2 = x)$.

Note: order of $\forall$ & $\exists$ matters! e.g.

$$\left.\begin{array}{l} \forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y^3 = x \\ \exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ y^3 = x \end{array}\right\} \text{ very different!}$$

§ 2.8 More on Conditional Statements

Suppose open sentences $T(x)$: x is a multiple of 10
$F(x)$: x is a multiple of 5

$T(x) \Rightarrow F(x)$ is true statement because $\underbrace{\forall x \in \mathbb{Z}}_{\text{from context.}},\ T(x) \Rightarrow F(x)$

$x = 30$: $T \Rightarrow T$ ✓
$x = 35$: $F \Rightarrow T$ ✓
$x = 38$: $F \Rightarrow F$ ✓

¿ $F(x) \Rightarrow T(x)$ is sometimes true, sometimes false → open sentence ?

$x = 30$: $T \Rightarrow T$ ✓
$x = 35$: $T \Rightarrow F$
$x = 38$: $F \Rightarrow F$ ✓

Whenever we have two open sentences about objects $x \in X$, $F(x) \Rightarrow T(x)$ is understood to mean $\forall x \in X,\ F(x) \Rightarrow T(x)$.

Hence this is a <u>false statement.</u>

Def (new, more general):

Given P, Q statements/open sentences (regardless)

$P \Rightarrow Q$ is a statement &

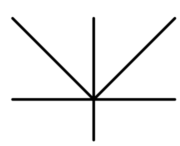
True if impossible for P true when Q false,

False if at least one case P true Q false.
(counter example)

ex. If f has a local minimum at a, then $f'(a) = 0$. False

- For <u>all</u> functions of a real variable

False if you can find a counter example $f(x) = |x|$



ex. If f has a local minimum at a and $f'(a)$ exists, then $f'(a) = 0$

- impossible for f to have a local min at a when $f'(a)$ exists & is nonzero. True

§2.9 Translating English to Symbolic Logic

e.g. Goldbach's Conjecture:

Every even integer greater than 2 is the sum of 2 prime numbers.

Let $P = \{2, 3, 5, 7, 11, \dots\}$
$X = \{4, 6, 8, \dots\}$

SAME
$$\begin{cases} \forall x \in X, \ \exists p, q \in P, \ p + q = x. \\ x \in X \Rightarrow \exists p, q \in P, \ p + q = x. \end{cases}$$

Fact 2.2 Suppose X is a set and $Q(x)$ is a statement about x for each $x \in X$. The following statements mean the same thing:

$\forall x \in X, Q(x)$
$(x \in X) \Rightarrow Q(x)$.

ex. $\lim_{x \to a} f(x) = L$: For any positive number ε there exists a positive number δ such that $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$.

: $\forall \varepsilon > 0, \ \exists \delta > 0, \ |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

ex. There is no largest prime number : For every prime number p, there is a prime number larger than p.

$\forall p \in P, \ \exists q \in P, \ q > p$

P = set of prime #s $= \{2, 3, 5, 7, 11, \dots\}$

7. There exists a real number a for which $a + x = x$ for every real number x.

$\exists a \in \mathbb{R}, \ \forall x \in \mathbb{R}, \ a + x = x.$ ✓

What's the difference?

$\forall x \in \mathbb{R}, \ \exists a \in \mathbb{R}, \ a + x = x.$ ✗

SOME NON-MATH EXAMPLES

ex. EVERYBODY IN THE DORM HAS A ROOMMATE THEY DON'T LIKE.

DEFINE SET $D = \{ p : p \text{ LIVES IN THE DORM} \}$
OPEN SENTENCE $R(x,y)$ = x AND y ARE ROOMMATES
OPEN SENTENCE $L(x,y)$ = x LIKES y

$\forall x \in D, \exists y, R(x,y) \wedge \sim L(x,y)$

Example 2.1.4. What do the following statements mean? Are they true or false? The universe of discourse in each case is $\mathbb{N}$, the set of all natural numbers.

1. $\forall x \exists y (x < y)$.
2. $\exists y \forall x (x < y)$.
3. $\exists x \forall y (x < y)$.
4. $\forall y \exists x (x < y)$.
5. $\exists x \exists y (x < y)$.
6. $\forall x \forall y (x < y)$.

§2.10 NEGATING STATEMENTS

PROVING THAT P IS TRUE IS THE SAME AS PROVING $\sim P$ IS FALSE (& VICE VERSA).

RECALL DE MORGAN'S LAWS:

$$\boxed{\begin{aligned} \sim(P \wedge Q) &= (\sim P) \vee (\sim Q) \\ \sim(P \vee Q) &= (\sim P) \wedge (\sim Q) \end{aligned}}$$

NEGATING QUANTIFIED STATEMENTS:

$\sim(\forall x \in X, P(x))$: IT IS NOT THE CASE THAT FOR ALL x IN X, $P(x)$ IS TRUE.

: $P(x)$ IS NOT TRUE FOR SOME $x \in X$.

: $\exists x \in X, \sim P(x)$.

$\sim(\exists x \in X, P(x))$: IT IS NOT THE CASE THAT THERE EXISTS AN $x \in X$ SUCH THAT $P(x)$ IS TRUE.

: $P(x)$ IS NOT TRUE FOR ALL $x \in X$.

: $\forall x \in X, \sim P(x)$

Summary Negating a quantified statement = ① Switch quantifier $\forall \leftrightarrow \exists$
② Negate the rest (open sentence)

$$\sim(\forall x \in X,\ P(x)) = \exists x \in X,\ \sim P(x)$$
$$\sim(\exists x \in X,\ P(x)) = \forall x \in X,\ \sim P(x)$$

ex. $\sim(\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ xy = 1) = \exists x \in \mathbb{R},\ \sim(\exists y \in \mathbb{R},\ xy = 1)$

$\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \sim(xy = 1)$

$\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ xy \neq 1.$

ex. Negate: **3.** For every prime number p, there is another prime number q with $q > p$.

Negating Conditional Statements

$$\boxed{\sim(P \Rightarrow Q) = P \wedge \sim Q}$$

P	Q	$\sim Q$	$P \Rightarrow Q$	$\sim(P \Rightarrow Q)$	$P \wedge \sim Q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

✓

ex. Negate: $3^2 = 9 \Rightarrow \sqrt{9} = 3$ T ⇒ T : T ✓

ex. Negate: $(-3)^2 = 9 \Rightarrow \sqrt{9} = -3$ T ⇒ F : F ⊗

ex. Negate: $x^2 = 9 \Rightarrow \sqrt{9} = x$, i.e. $\forall x \in \mathbb{R},\ x^2 = 9 \Rightarrow \sqrt{9} = x$

ex. What does it mean to say that $\lim_{x \to a} f(x) \neq L$?

Please read §2.11-12