§3.5 Counting Subsets

How many subsets of size k are there of a set of size n?

Def: $\binom{n}{k}$ or $C(n,k)$ or ${}_nC_k$ "n choose k"

denotes the # of subsets that can be made by choosing k elements from a set with n elements.

Note: $\binom{n}{k} = 0$ when $k > n$

Thm: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, $0 \leq k \leq n$

Proof: A list of k elements without repetition can be made from n elements in $P(n,k)$ ways.

Such a list can be made in 2 steps: (1) choose of subset of k elements from n elements
(2) Permute the k elements (choose order)

By multiplication principal, we have

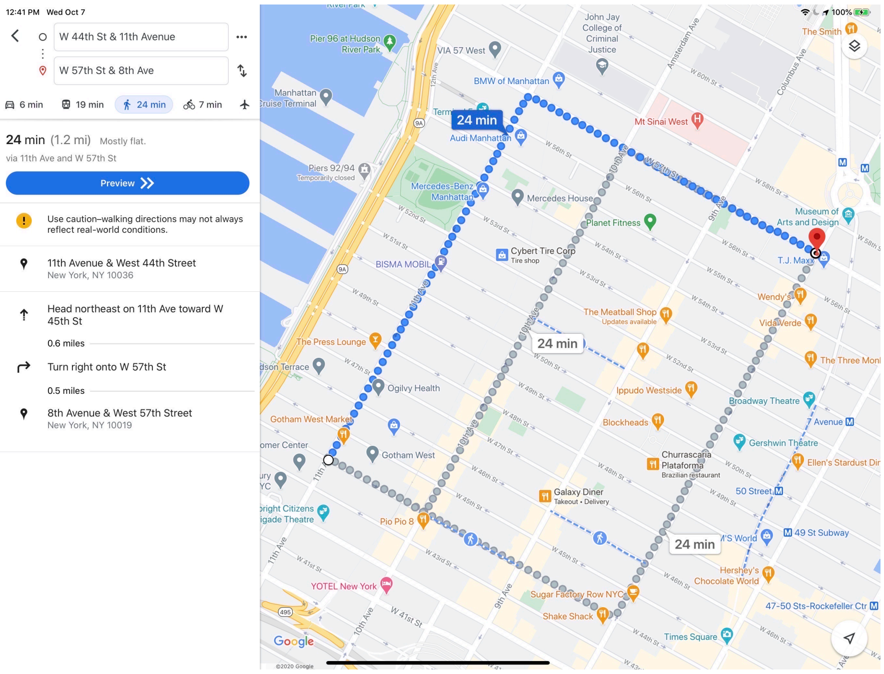
$$P(n,k) = \underbrace{\boxed{\binom{n}{k}}}_{(1)} \times \underbrace{\boxed{k!}}_{(2)} \Rightarrow \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!} \checkmark$$

ex. The fish section of a pet store is stocked with 8 guppies, 6 angelfish, 13 goldfish, and 9 rainbowfish.

1. How many ways are there to select one of each?
2. How many ways are there to select two of each?

5. How many 16-digit binary strings contain exactly seven 1's? (Examples of such strings include 0111000011110000 and 0011001100110010, etc.)

ex. How many ways are there to walk from corner of W 44 St and 11th Ave to the corner of W 57 St and 8th Ave, assuming you do not go out of your way (only walk north and east)?



Note: You must walk 3 blocks East & 13 blocks North.

e.g. NNNE NNNNE NNNNNN

$$\boxed{\textbf{Thm} \quad \binom{n}{k} = \binom{n}{n-k}}$$

Proof: Let $\mathcal{U}$ be a set, $|\mathcal{U}| = n$.

For every subset $X \subseteq \mathcal{U}$ with size k, $\overline{X}$ is a subset with size $n-k$.

$$\therefore \left| \{ X \subseteq \mathcal{U} : |X| = k \} \right| = \left| \{ X \subseteq \mathcal{U} : \overline{X} = n-k \} \right|$$

That is, $\binom{n}{k} = \binom{n}{n-k}$ ✓

Alt Proof:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-(n-k))!\,(n-k)!} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \binom{n}{n-k}$$

✓

§3.6 Pascal's Triangle & the Binomial Thm

Thm $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Proof: (Combinatorial proof : both sides answer same counting question. Read §3.10)

$$\#\text{ subsets of size } k \text{ of } \{1,2,3,\dots,n,n+1\} = \#\text{ subsets of size } k \text{ that include } n+1 + \#\text{ subsets of size } k \text{ that don't include } n+1$$

(choose other $k-1$)

$$= \binom{n}{k-1} + \binom{n}{k} \quad ✓$$

Alt: $$\binom{n}{k-1} + \binom{n}{k} = \frac{k}{k}\cdot\frac{n!}{(k-1)!\,(n-k+1)!} + \frac{(n-k+1)}{(n-k+1)}\cdot\frac{n!}{k!\,(n-k)!}$$

$$\text{LCD} = k!\,(n-k+1)!$$

$$= \frac{kn! + n!(n-k+1)}{k!\,(n-k+1)!} = \frac{n!\,(\cancel{k} + n - \cancel{k} + 1)}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k} \quad ✓$$

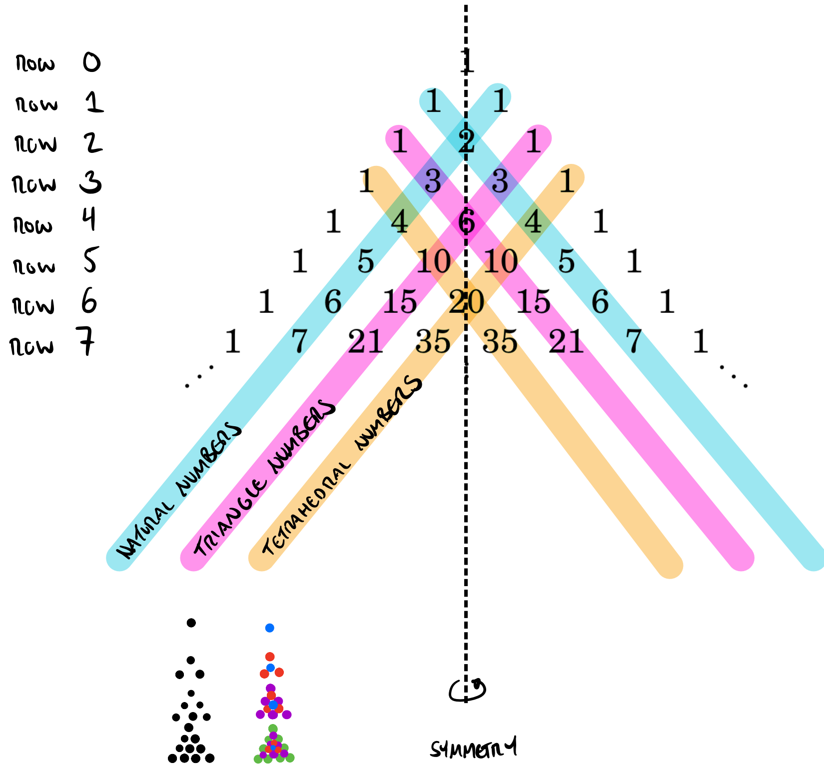
$\binom{n}{k}$	…	k=-1	k=0	k=1	k=2	k=3	k=4	k=5	k=6	…
n=0		0	1	0	0	0	0	0	0	
n=1		0	1	1	0	0	0	0	0	
n=2		0	1	2	1	0	0	0	0	
n=3		0	1	3	3	1	0	0	0	
n=4		0	1	4	6	4	1	0	0	
n=5		0	1	5	10	10	5	1	0	
n=6		0	1	6	15	20	15	6	1	

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\vdots$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
⋮

Figure 3.3. Pascal's triangle

Blaise Pascal (1623-1662)



MANY MORE PATTERNS EXIST!

https://www.cut-the-knot.org/arithmetic/combinatorics/PascalTriangleProperties.shtml

Theorem 3.1 **(Binomial Theorem)** If n is a non-negative integer, then
$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$.

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

COEFFICIENTS ARE ENTRIES IN n^{th} ROW OF PASCALS TRIANGLE

ex. $(x^2+3)^5 = 1(x^2)^5 + 5(x^2)^4 3 + 10(x^2)^3 3^2 + 10(x^2)^2 3^3 + 5(x^2)3^4 + 1\cdot 3^5$

$= x^{10} + 15x^8 + 90x^6 + 270x^4 + 405x^2 + 243$

0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

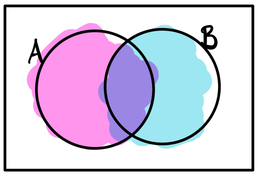
7. Use the binomial theorem to show $\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n$.

BINOMIAL THM: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

HERE, $\sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 3^k = (1+3)^n = 4^n$ ✓

§3.7 The Inclusion-Exclusion Principle

Fact 3.6 Inclusion-Exclusion Formula
If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.



7. Consider 4-card hands dealt off of a standard 52-card deck. How many hands are there for which all 4 cards are of the same suit or all 4 cards are red?

Let S = set of all 4-card hands, all same suit.
R = set of all 4-card hands, all red.

$$|S \cup R| = |S| + |R| - |S \cap R|$$
$$= 4\binom{13}{4} + \binom{26}{4} - 2\binom{13}{4}$$

§3.8 Multisets

Def: A multiset is collection of objects (like a set!) where elements can be repeated. Order doesn't matter.

Notation: $M = [1, 2, 2, 3, 3, 3]$ is a multiset with 6 elements. $|M| = 6$

$= [3, 2, 1, 3, 2, 3]$ is the same multiset.

$1 \in M$ has multiplicity 1
$2 \in M$ has multiplicity 2
$3 \in M$ has multiplicity 3

1. How many 10-element multisets can be made from the symbols $\{1,2,3,4\}$?

ORDER DOESN'T MATTER, JUST $\underbrace{\#\,1\text{'s},\ \#\,2\text{'s},\ \#\,3\text{'s},\ \#\,4\text{'s}}_{\text{ADD TO } 10}$

COMBINATORIAL ARGUMENT: # MULTISETS IS THE SAME AS # SOMETHING EASIER TO COUNT (READ §3.10)

PICTURE 10 ELEMENTS AS 10 $\ast$'s: $\ast\ \ast\ \ast\ \ast\ \ast\ \ast\ \ast\ \ast\ \ast\ \ast$

INSERT 3 BARS TO SEPARATE 1's, 2's, 3's, & 4's

e.g. $\ast\ \ast\ \ast \mid \ast\ \ast\ \ast\ \ast \mid \ast \mid \ast\ \ast \rightarrow [1,1,1,2,2,2,2,3,4,4]$

e.g. $\ast\ \ast\ \ast\ \ast \mid\mid \ast\ \ast\ \ast\ \ast\ \ast \mid \ast \rightarrow [1,1,1,1,3,3,3,3,3,4]$

NO 2's

e.g. $\mid\mid \ast\ \ast\ \ast\ \ast\ \ast\ \ast\ \ast\ \ast\ \ast\ \ast \mid \rightarrow [3,3,3,3,3,3,3,3,3,3]$

NO 1's, NO 2's, NO 4's

THE 10 STARS + 3 BARS YIELD 13 POSITIONS FOR SYMBOLS (STARS & BARS)
A MULTISET IS DETERMINED BY CHOOSING WHICH 3 OF 13 POSITIONS FOR BARS.

ANSWER: $\binom{13}{3} = \dfrac{13!}{3!\,10!} = \dfrac{13\cdot 12\cdot 11}{3\cdot 2} = 13\cdot 2\cdot 11 = 286$

Fact 3.7 The number of k-element multisets that can be made from the elements of an n-element set $X = \{x_1, x_2, \ldots, x_n\}$ is

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}.$$

This works because any cardinality-k multiset made from the n elements of X can be encoded in a star-and-bar list of length $k+n-1$, having form

$$\overbrace{\ast\ \ast\ \ast\ \ast\ \ast}^{\ast \text{ for each } x_1} \mid \overbrace{\ast\ \ast\ \ast\ \ast\ \ast}^{\ast \text{ for each } x_2} \mid \overbrace{\ast\ \ast\ \ast\ \ast\ \ast}^{\ast \text{ for each } x_3} \mid \cdots\cdots \mid \overbrace{\ast\ \ast\ \ast\ \ast\ \ast}^{\ast \text{ for each } x_n}$$

with k stars and $n-1$ bars separating the n groupings of stars. Such a list can be made by selecting $n-1$ positions for the bars, and filling the remaining positions with stars, and there are $\binom{k+n-1}{n-1}$ ways to do this.

7. In how many ways can you place 20 identical balls into five different boxes?

9. A bag contains 50 pennies, 50 nickels, 50 dimes and 50 quarters. You reach in and grab 30 coins. How many different outcomes are possible?

§3.9 DIVISION & PIGEONHOLE PRINCIPLE

FLOOR FUNCTION $\lfloor x \rfloor$ = ROUND DOWN TO NEAREST INTEGER = $\max(\{n \in \mathbb{Z} : n \leq x\})$

CEILING FUNCTION $\lceil x \rceil$: ROUND UP TO NEAREST INTEGER = $\min(\{n \in \mathbb{Z} : n \geq x\})$

e.g. $\lfloor \pi \rfloor = 3$, $\lceil \pi \rceil = 4$, $\lfloor 5 \rfloor = \lceil 5 \rceil = 5$

$\lfloor 6.9 \rfloor = 6$, $\lceil 8.1 \rceil = 9$

Fact 3.9 **(Division Principle)**
Suppose n objects are placed into k boxes.
Then at least one box contains $\lceil \frac{n}{k} \rceil$ or more objects,
and at least one box contains $\lfloor \frac{n}{k} \rfloor$ or fewer objects.

NOTE: AVERAGE $\frac{n}{k}$ OBJECTS PER BOX.

- CANNOT HAVE ALL BOXES WITH FEWER
- CANNOT HAVE ALL BOXES WITH MORE

IN PARTICULAR

Fact 3.10 **(Pigeonhole Principle)**
Suppose n objects are placed into k boxes.
If $n > k$, then at least one box contains more than one object.
If $n < k$, then at least one box is empty.

1. Show that if six integers are chosen at random, then at least two of them will have the same remainder when divided by 5.

3. What is the fewest number of times you must roll a six-sided dice before you can be assured that 10 or more of the rolls resulted in the same number?