Ch. 6 Proof by Contradiction

ex. **Proposition.** There are no integers $a, b \in \mathbb{Z}$ such that $51a + 87b = 1$.

True / False — The statement must be one or the other.

Proof by Contradiction:

1. Assume the statement is false.
2. Form logical argument to conclude something known to be false (contradiction)
3. The assumption cannot be false. Therefore assumption is true.

Proof.

1. Assume there are integers a & b such that $51a + 87b = 1$.
2. Then $3(17a + 29b) = 1$ and $17a + 29b = \frac{1}{3}$. Since $17a + 29b \in \mathbb{Z}$, it follows that $\frac{1}{3} \in \mathbb{Z}$.
3. This contradicts the fact that $\frac{1}{3}$ is not an integer. Therefore, our assumption that are integers $a, b \in \mathbb{Z}$ such that $51a + 87b = 1$ must be false. ■

P & $\sim P \Rightarrow \overbrace{(Q \wedge \sim Q)}^{\text{contradiction : never true}}$ are logically equivalent

$\Rightarrow\!\!\!\!\times\!\!\!\!\Leftarrow$

contrapositive

$\sim(Q \wedge \sim Q) \Rightarrow \sim(\sim P)$

$\underbrace{(\sim Q \vee Q)}_{\text{always true}} \Rightarrow P$

P	Q	$\sim P$	$\sim Q$	$Q \wedge \sim Q$	$\sim P \Rightarrow (Q \wedge \sim Q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	F
F	F	T	T	F	F

✓

Def: $x \in \mathbb{R}$ is rational if $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$.

The set of rational numbers is

$$\mathbb{Q} = \left\{ x \in \mathbb{R} : x = \frac{a}{b}, \text{ with } a, b \in \mathbb{Z} \right\}.$$

$x \in \mathbb{R}$ is irrational if $x \neq \frac{a}{b}$ for any $a, b \in \mathbb{Z}$

ex. **Theorem.** $\sqrt{2}$ is irrational.

Proof. Assume, for the sake of contradiction, that $\sqrt{2} \in \mathbb{Q}$.
By definition, $\sqrt{2} = \frac{a}{b}$.

Without loss of generality, let $\frac{a}{b}$ be reduced.
In particular, a & b are not both even.

Then $\sqrt{2} = \frac{a}{b} \Rightarrow \sqrt{2}\, b = a \Rightarrow 2b^2 = a^2$.
Thus a^2 is even, and since the product of two odd numbers is odd,
it must be that a is even. Let $a = 2x$, $x \in \mathbb{Z}$.

Then $2b^2 = (2x)^2 = 4x^2 \Rightarrow b^2 = 2x^2$.
Thus b^2 is even, and so b must be even.
Let $b = 2y$, $y \in \mathbb{Z}$.

Therefore both a & b are even, and $\frac{a}{b}$ is not reduced.
This contradicts the fact that $\frac{a}{b}$ is reduced.

Thus, our assumption that $\sqrt{2} \in \mathbb{Q}$ must be false.
That is, $\sqrt{2}$ is irrational. ∎

ex. **Theorem.** There are infinitely many prime numbers.

Proof. Assume, for the sake of contradiction, that there are finitely many prime numbers $p_1, p_2, \dots, p_n \in \mathbb{N}$.

Set $a = p_1 p_2 \dots p_n + 1 \in \mathbb{N}$.

Like all natural numbers greater than 1, a has at least 1 prime divisor, say p_k.
Then $a = p_k x$ for some $x \in \mathbb{N}$.

We have $p_k x = p_1 p_2 \cdots p_{k-1} p_k p_{k+1} \cdots p_n + 1$

$\Rightarrow \quad x = p_1 p_2 \cdots p_{k-1} p_{k+1} \cdots p_n + \frac{1}{p_k}$

$\Rightarrow \quad \frac{1}{p_k} = x - p_1 p_2 \cdots p_{k-1} p_{k+1} \cdots p_n \in \mathbb{Z}$.

But $\frac{1}{p_k}$ cannot be an integer. $\Rightarrow\!\Leftarrow$ ∎

PROVING CONDITIONAL STATEMENTS BY CONTRADICTION

PROP: $P \Rightarrow Q$.

PROOF: ASSUME $\sim(P \Rightarrow Q)$, i.e. $P \wedge \sim Q$.

$\vdots$

$\Rightarrow\Leftarrow$ ■

ex. **16.** If a and b are positive real numbers, then $a+b \geq 2\sqrt{ab}$.

PROOF. (CONTRADICTION) ASSUME a, b POSITIVE REAL NUMBERS & $a+b < 2\sqrt{ab}$.

THEN $\sqrt{a}^2 - 2\sqrt{a}\sqrt{b} + \sqrt{b}^2 < 0$

THAT IS, $(\sqrt{a} - \sqrt{b})^2 < 0$.

$\Rightarrow\Leftarrow$.

ex. **9.** Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

PROOF: (CONTRADICTION) ASSUME $a \in \mathbb{Q}$ AND $ab \notin \mathbb{Q}$ AND $b \in \mathbb{Q}$.

BY DEFINITION, $a = \frac{p}{q}$ AND $b = \frac{m}{n}$ FOR SOME $p, q, m, n \in \mathbb{Z}$.

THEN $ab = \frac{p}{q} \cdot \frac{m}{n} = \frac{pm}{qn}$.

SINCE $pm, qn \in \mathbb{Z}$, WE HAVE $ab \in \mathbb{Q}$ $\Rightarrow\Leftarrow$ ■