Ch. 8 Proofs Involving Sets

(1) $a \in A$
(2) $A \subseteq B$
(3) $A = B$
((4) Perfect Numbers)

Review of Sets:

- $A \times B = \{(x,y) : x \in A, y \in B\}$
- $A \cup B = \{x : (x \in A) \vee (x \in B)\}$
- $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$
- $A - B = \{x : (x \in A) \wedge (x \notin B)\}$
- $\bar{A} = \mathcal{U} - A$
- $A \subseteq B$ means $(x \in A) \Rightarrow (x \in B)$, i.e. $\forall x \in A, x \in B$
- $\mathcal{P}(A) = \{X : X \subseteq A\}$

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§8.1 How to Prove $a \in A$

Set-Builder Notation	$A = \{x : P(x)\}$	$A = \{x \in S : P(x)\}$
How to show $a \in A$	Show $P(a)$ is true	Show $a \in S$ & $P(a)$ is true

ex. Let $A = \{x \in \mathbb{Z} : 2 \mid x$ and $3 \mid x\}$.
Prove that $18 \in A$.

Proof: 18 is an integer and since $18 = 2 \cdot 9 = 3 \cdot 6$, it follows that $2 \mid 18$ and $3 \mid 18$. Therefore $18 \in A$.

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§ 8.2 How to Prove $A \subseteq B$

$(a \in A) \Rightarrow (a \in B)$

(1) Direct
(2) Contrapositive
(3) Contradiction

ex. Proposition: $A \subseteq A \cup B$.

Proof: (Direct) Assume $x \in A$.

Then $(x \in A) \vee (x \in B)$ is a true statement.

Thus $x \in \{a : (a \in A) \vee (a \in B)\} = A \cup B$.

Therefore, $(x \in A) \Rightarrow (x \in A \cup B)$.

That is, $A \subseteq A \cup B$. $\blacksquare$

ex. **25.** Suppose A, B, C and D are sets. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Proof: (Direct) Assume $(x,y) \in (A \times B) \cup (C \times D)$.

Then $(x,y) \in A \times B$ or $(x,y) \in C \times D$.

If $(x,y) \in A \times B = \{(a,b) : a \in A, b \in B\}$, then $x \in A$ and $y \in B$.

Since $A \subseteq A \cup C$ and $B \subseteq B \cup D$, it follows that

$x \in A \cup C$ and $y \in B \cup D$.

Thus $(x,y) \in \{(a,b) : a \in A \cup C, b \in B \cup D\} = (A \cup C) \times (B \cup D)$.

Similarly, if $(x,y) \in C \times D$ then $(x,y) \in (A \cup C) \times (B \cup D)$.

Therefore $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. $\blacksquare$

§8.3 How to Prove A = B

2 sets are equal when they contain the exact same elements.

$(A = B) \Leftrightarrow (A \subseteq B) \wedge (B \subseteq C)$.

ex. Proposition: $\{x \in \mathbb{Z} : 15 \mid x\} = \{x \in \mathbb{Z} : 3 \mid x\} \cap \{x \in \mathbb{Z} : 5 \mid x\}$

Proof:

First, assume $a \in \{x \in \mathbb{Z} : 15 \mid x\}$.
Since $15 \mid a$ we have $a = 15n$ for some $n \in \mathbb{Z}$.
Then $a = 3(5n) = 5(3n)$, with $5n, 3n \in \mathbb{Z}$.
Therefore $3 \mid a$ and $5 \mid a$.
It follows that $a \in \{x \in \mathbb{Z} : 3 \mid x\}$ and $a \in \{x \in \mathbb{Z} : 5 \mid x\}$.
Therefore $a \in \{x \in \mathbb{Z} : 3 \mid x\} \cap \{x \in \mathbb{Z} : 5 \mid x\}$.

Next, assume $a \in \{x \in \mathbb{Z} : 3 \mid x\} \cap \{x \in \mathbb{Z} : 5 \mid x\}$.
It follows that $a = 3m = 5n$ for some $m, n \in \mathbb{Z}$.
Then $5a = 15m$ and $3a = 15n$, and so

$$5a - 3a = 15(m-n)$$
$$2a = 15(m-n).$$

Therefore $15(m-n)$ is even.
Since 15 is odd, it must be that $m-n$ is even.
So $m - n = 2c$ for some $c \in \mathbb{Z}$.
Now
$$2a = 15 \cdot 2c$$
$$a = 15c.$$

Therefore $15 \mid a$ & $a \in \{x \in \mathbb{Z} : 15 \mid x\}$.

This proves the proposition. ∎

ex. **23.** For each $a \in \mathbb{R}$, let $A_a = \{(x, a(x^2-1)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Prove that $\bigcap_{a\in\mathbb{R}} A_a = \{(-1,0),(1,0)\}$.

PROOF: FIRST WE SHOW $\{(-1,0),(1,0)\} \subseteq \bigcap_{a\in\mathbb{R}} A_a$.

WE DO SO DIRECTLY BY SHOWING

$$(x,y) \in \{(-1,0),(1,0)\} \Rightarrow x \in \bigcap_{a\in\mathbb{R}} A_a.$$

ASSUME $(x,y) \in \{(-1,0),(1,0)\}$.

THEN $x=1$ OR $x=-1$. EITHER WAY, $x \in \mathbb{R}$. FURTHERMORE, $x^2=1$.

THUS, $y=0=a\cdot 0 = a(x^2-1)$ FOR ANY REAL NUMBER a.

THAT IS, $(x,y) \in \bigcap_{a\in\mathbb{R}} \{(x, a(x^2-1)) : x \in \mathbb{R}\}$

NEXT WE SHOW $\bigcap_{a\in\mathbb{R}} A_a \subseteq \{(-1,0),(1,0)\}$.

WE DO SO BY PROVING THE CONTRAPOSITIVE STATEMENT:

IF $(x,y) \notin \{(-1,0),(1,0)\}$ THEN $(x,y) \notin \bigcap_{a\in\mathbb{R}} A_a$.

CASE 1: $x \notin \{-1,1\}$. THEN $x^2-1 \neq 0$.

SET $a_0 = \dfrac{y+1}{x^2-1} \in \mathbb{R}$.

THEN $a_0(x^2-1) = \dfrac{y+1}{x^2-1}(x^2-1) = y+1 \neq y$

THUS $(x,y) \notin \{(x, a_0(x^2-1)) : x \in \mathbb{R}\}$

THEREFORE $(x,y) \notin \bigcap_{a\in\mathbb{R}} A_a = A_{a_0}$

CASE 2: $x \in \{-1,1\}$ AND $y \neq 0$.

IN THIS CASE, $x^2-1=0 \neq y$.

THUS $(x,y) \notin \{(x, x^2-1) : x \in \mathbb{R}\} = A_1$.

THEREFORE $(x,y) \notin \bigcap_{a\in\mathbb{R}} A_a$.

THEREFORE $\bigcap_{a\in\mathbb{R}} \{(x, a(x^2-1)) : x \in \mathbb{R}\} = \{(-1,0),(1,0)\}$. ∎

MORE GENERAL PROPERTIES OF SETS, WHEN VERY LITTLE OR NO INFORMATION ABOUT THE SETS IS KNOWN, CAN BE PROVED USING SET DEFINITIONS AND LOGICALLY EQUIVALENT STATEMENTS.

ex. PROVE **DEMORGAN'S LAW:** IF A & B ARE SUBSETS OF SOME UNIVERSAL SET $\mathcal{U}$, THEN

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

PROOF: ASSUME $A, B \subseteq \mathcal{U}$.

THEN $\overline{A \cup B} = \mathcal{U} - (A \cup B)$

$$= \{x : (x \in \mathcal{U}) \wedge \sim(x \in A \cup B)\}$$

$$= \{x : (x \in \mathcal{U}) \wedge \sim(x \in A \vee x \in B)\}$$

$$= \{x : (x \in \mathcal{U}) \wedge (\sim(x \in A) \wedge \sim(x \in B))\}$$

$$= \{x : (x \in \mathcal{U}) \wedge \sim(x \in A) \wedge (x \in \mathcal{U}) \wedge \sim(x \in B)\}$$

$$= \{x : ((x \in \mathcal{U}) \wedge \sim(x \in A)) \wedge ((x \in \mathcal{U}) \wedge \sim(x \in B))\}$$

$$= \{x : (x \in \mathcal{U}) \wedge \sim(x \in A)\} \cap \{x : (x \in \mathcal{U}) \wedge \sim(x \in B)\}$$

$$= (\mathcal{U} - A) \cap (\mathcal{U} - B)$$

$$= \overline{A} \cap \overline{B}. \quad \blacksquare$$

ex. USE DEMORGAN'S LAW $\overline{A \cup B} = \overline{A} \cap \overline{B}$ TO REWRITE

(a) $A \cap B = \overline{\overline{A}} \cap \overline{\overline{B}} = \overline{\overline{A} \cup \overline{B}}$

(b) $A \cup B = \overline{\overline{A \cup B}} = \overline{\overline{A} \cap \overline{B}}$

(c) $A - B = A \cap \overline{B} = \overline{\overline{A} \cup B}$

PLEASE READ §8.4.
YOU WILL NOT BE TESTED ON ITS CONTENTS
BUT IT IS COOL!