Ch 9. Disproof

Categorizing Statements:

True	Unknown	False
Theorems & Propositions	Conjectures: statements with truth/falsity unknown	False Statements

Unknown $\xrightarrow{\text{Proof}}$ True

Unknown $\xrightarrow{\text{Disproof}}$ False

$\boxed{\text{To disprove } P\text{, we must prove } \sim P}$ using direct, contrapositive, contradiction.

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§9.1 Counterexamples

Universal Statements

$$\boxed{\text{To prove } Q: \forall x \in S, P(x)\text{, we prove } \sim Q: \sim(\forall x \in S, P(x)) = \underbrace{\exists x \in S}, \sim P(x).}$$

This only requires us to find one such x, called a counterexample.

ex. Prove or Disprove: For all positive real numbers a & b, $a+b < 2\sqrt{ab}$.

(Universal statement: disprove with counterexample.)

The universal statement is false.
As a counterexample, take $a = 18$ & $b = 2$.
Then $a+b = 20$ and $2\sqrt{ab} = 12$,
So it is not true that $a+b < 2\sqrt{ab}$ for all positive real numbers a & b. ■

Note: You are responsible for showing why your counterexample is indeed a counterexample!

CONDITIONAL STATEMENTS

RECALL: $P(x) \Rightarrow Q(x) = \forall x \in \{x : P(x)\}, Q(x)$

$$\sim(P(x) \Rightarrow Q(x)) = \sim(\forall x \in \{x : P(x)\}, Q(x))$$

$$= \exists x \in \{x : P(x)\}, \sim Q(x)$$

TO DISPROVE $P(x) \Rightarrow Q(x)$, ONE COUNTEREXAMPLE x SUCH THAT $P(x) \wedge \sim Q(x)$ IS TRUE IS SUFFICIENT.

ex. PROVE OR DISPROVE:

12. If $a, b, c \in \mathbb{N}$ and ab, bc and ac all have the same parity, then a, b and c all have the same parity.

THE STATEMENT IS FALSE. AS A COUNTEREXAMPLE, LET $a = 1$, $b = c = 2$.
THEN $ab = (1)(2) = 2$, $bc = (2)(2) = 4$, AND $ac = (1)(2) = 2$ ALL ALL EVEN.
SO ab, bc, AND ac ALL HAVE SAME PARITY, BUT a IS ODD WHILE b & c ARE EVEN.

§9.2 DISPROVING EXISTENCE STATEMENTS

TO DISPROVE $Q: \exists x \in S, P(x)$,
WE MUST PROVE $\sim Q : \sim(\exists x \in S, P(x)) = \forall x \in S, \sim P(x) = \underbrace{x \in S \Rightarrow \sim P(x)}$.

COUNTEREXAMPLE NOT SUFFICIENT

CONDITIONAL STATEMENT
1. DIRECT
2. CONTRAPOSITIVE
3. CONTRADICTION

ex. DISPROVE THE STATEMENT:
THERE EXIST DISTINCT PRIME NUMBERS a & b SUCH THAT $a + b > ab$.

WE DISPROVE THE STATEMENT BY PROVING ITS NEGATION:
IF a, b ARE DISTINCT PRIME NUMBERS THEN $a + b < ab$.
WE DO SO DIRECTLY.

ASSUME a, b ARE PRIME & WLOG ASSUME $a < b$.
THEN $a + b < b + b = 2b \leq ab$, WHERE THE LAST INEQUALITY FOLLOWS FROM THE FACT THAT ALL PRIMES ARE AT LEAST 2.
THUS $a + b < ab$. ∎

§9.3 DISPROOF BY CONTRADICTION

TO DISPROVE P, ASSUME P IS TRUE & SHOW THIS LEADS TO A CONTRADICTION.

ex. **21.** There exist prime numbers p and q for which $p - q = 97$.

WE DISPROVE BY CONTRADICTION.

ASSUME p & q ARE PRIME AND $p - q = 97$, i.e. $p = q + 97$.

(CASE 1) IF $q = 2$ THEN $p = 99 = 9 \cdot 11$ IS NOT PRIME $\Rightarrow\!\Leftarrow$

(CASE 2) IF $q \neq 2$ THEN q IS ODD BECAUSE 2 IS THE ONLY EVEN PRIME.

SINCE $97 = 2 \cdot 48 + 1$ IS ODD, IT FOLLOWS THAT p IS EVEN, AS IT IS THE SUM OF TWO ODD NUMBERS.

THEREFORE $p = 2$. THIS IS A CONTRADICTION BECAUSE $p = q + 97$ & q PRIME IMPLIES $p \geq 99$. $\Rightarrow\!\Leftarrow$

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