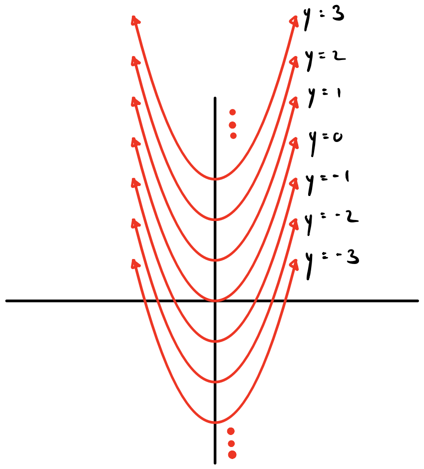
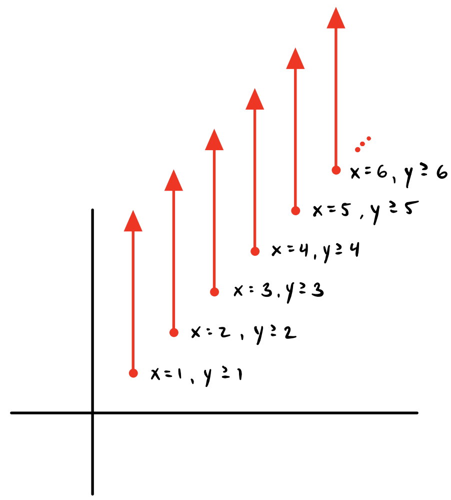
Exam 1

1. Sketch the following set of points in the xy-plane.
   (a) (6 points) $\{(x, x^2 + y) \ : \ x \in \mathbb{R}, y \in \mathbb{Z}\}$
   (b) (6 points) $\{(x, y) \in \mathbb{N} \times \mathbb{R} \ : \ y \geq x\}$

(a)



(b)



2. Write the following sets in set-builder notation.
   (a) (6 points) $\{\ldots, \frac{5}{8}, \frac{5}{4}, \frac{5}{2}, 5, 10, 20, 40, 80, \ldots\}$
   (b) (6 points) $\left\{\frac{2}{7}, \frac{4}{11}, \frac{6}{15}, \frac{8}{19}, \ldots\right\}$
   (c) (6 points) $\{\{3\}, \{3, 6\}, \{3, 6, 9\}, \{3, 6, 9, 12\}, \{3, 6, 9, 12, 15\}, \ldots\}$

(a) $\{5 \cdot 2^n : n \in \mathbb{Z}\}$

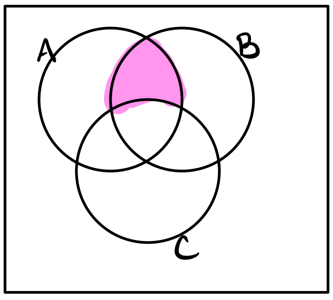
(b) $\left\{\frac{2n}{3+4n} : n \in \mathbb{N}\right\}$ or $\left\{\frac{2n}{7+4(n-1)} : n \in \mathbb{N}\right\}$

(c) $\{\{3k : k \in \mathbb{N}, 1 \leq k \leq n\} : n \in \mathbb{N}\}$ or $\{\{3, 6, 9, \ldots, 3n\} : n \in \mathbb{N}\}$

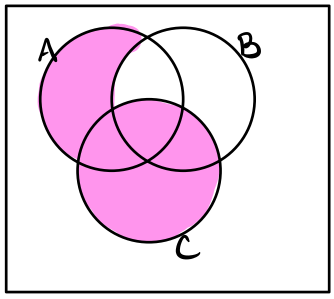
or $\left\{\bigcup_{k=1}^{n} \{3k\} : n \in \mathbb{N}\right\}$

3. Draw Venn diagrams for each of the following sets.
   (a) (6 points) $(A \cap B) - C$
   (b) (6 points) $(A - B) \cup C$

(a)



(b)



4. Write each of the following sets by listing its elements between braces or describing it with a familiar symbol or symbols (e.g. interval notation, set-builder notation, $\emptyset$, $\mathbb{Z}$, etc.)

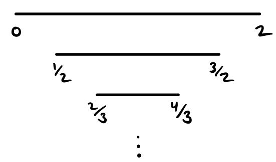
(a) (6 points) $\bigcup_{n\in\mathbb{N}} \left\{x \in \mathbb{R} \ : \ \frac{n-1}{n} < x < \frac{n+1}{n}\right\}$ $= \bigcup_{n\in\mathbb{N}} \left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$

(b) (6 points) $\bigcap_{n\in\mathbb{N}} \left\{x \in \mathbb{R} \ : \ \frac{n-1}{n} < x < \frac{n+1}{n}\right\}$ $= \bigcap_{n\in\mathbb{N}} \left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$

(c) (6 points) $\{X \subseteq \{a,b,c,d\} \ : \ |\mathscr{P}(X)| = 8\}$

(a) $(0,2) \cup \left(\frac{1}{2}, \frac{3}{2}\right) \cup \left(\frac{2}{3}, \frac{4}{3}\right) \cup \left(\frac{3}{4}, \frac{5}{4}\right) \cup \ldots$

$= (0,2)$



(b) $(0,2) \cap \left(\frac{1}{2}, \frac{3}{2}\right) \cap \left(\frac{2}{3}, \frac{4}{3}\right) \cap \left(\frac{3}{4}, \frac{5}{4}\right) \cap \ldots$

$= \{1\}$

(c) NOTE: $8 = |\mathscr{P}(X)| = 2^{|X|} \Rightarrow |X| = 3$

SO THIS IS THE SET OF ALL SUBSETS OF $\{a,b,c,d\}$ WITH 3 ELEMENTS.

THERE ARE $\binom{4}{3} = 4$ OF THEM.

$\{ \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\} \}$

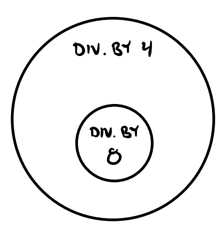
5. (6 points) Are the statements $P \Rightarrow Q$ and $(\sim P) \rightarrow (Q \wedge \sim Q)$ logically equivalent? Support your answer with a truth table.

P	Q	$\sim P$	$\sim Q$	$Q \wedge \sim Q$	$P \Rightarrow Q$	$(\sim P) \Rightarrow (Q \wedge \sim Q)$
T	T	F	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	F	T	F
F	F	T	T	F	T	F

NOT LOGICALLY EQUIVALENT

6. (6 points) Without changing its meaning, convert the following sentence into a sentence having the form *"If P then Q"*: An integer is divisible by 8 only if it is divisible by 4.

IF AN INTEGER IS DIVISIBLE BY 8, THEN IT IS DIVISIBLE BY 4.



7. Consider the following mathematical statement.

$$\forall n \in \mathbb{Z}, \exists X \subseteq \mathbb{N}, |X| = n$$

(a) (6 points) Rewrite the mathematical statement as an English sentence.

(b) (4 points) Is the mathematical statement true or false?

(c) (6 points) Write the negation of the mathematical statement in symbols. Then rewrite the negation of the mathematical statement as an English sentence.

(a) FOR ANY INTEGER n, THERE EXISTS A SUBSET X OF THE NATURAL NUMBERS WITH EXACTLY n ELEMENTS.

(b) FALSE : NO SET CAN HAVE A NEGATIVE NUMBER OF ELEMENTS.

(c) $\exists n \in \mathbb{Z}, \forall X \subseteq \mathbb{N}, |X| \neq n.$

THERE EXISTS AN INTEGER n SUCH THAT EVERY SUBSET X OF THE NATURAL NUMBERS DOES NOT CONTAIN n ELEMENTS.

OR, PHRASED DIFFERENTLY (SAME MEANING),

FOR SOME INTEGER n, EVERY SUBSET X OF NATURAL NUMBERS HAS A CARDINALITY THAT IS NOT EQUAL TO n.

8. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) (6 points) How many 6-element subsets of A contain exactly 2 even integers?

(b) (6 points) How many 6-element subsets of A contain exactly 2 even integers or contain both 8 and 9?

(a) CREATE A 6-ELEMENT SUBSET WITH EXACTLY 2 EVEN INTEGERS IN 2 STEPS :

(1) CHOOSE 2 EVEN INTEGERS FROM 2, 4, 6, 8

(2) CHOOSE 4 ODD INTEGERS FROM 1, 3, 5, 7, 9

READ 

\# WAYS TO DO THIS:

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4 \cdot 3}{2} = 6 \qquad \binom{5}{4} = \frac{5!}{4!\,1!} = 5$$

BY MULTIPLICATION PRINCIPLE, THERE ARE $6 \times 5 = 30$ SUCH SUBSETS.

(b) LET A = SET OF 6-ELEMENT SUBSETS WITH EXACTLY 2 EVEN INTEGERS
B = SET OF 6-ELEMENT SUBSETS THAT CONTAIN BOTH 8 & 9.

$|A| = 30$ (PART (a))

$|B| = \binom{7}{4} = \frac{7!}{4!\,3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35$

(8 & 9 MUST BE IN THE SET. OF THE REMAINING 7 ELEMENTS, CHOOSE 4.)

$|A \cap B| = \binom{3}{1} \times \binom{4}{3} = \frac{3!}{1!\,2!} \times \frac{4!}{3!\,1!} = 3 \times 4 = 12$

(8 (EVEN) IS IN THE SET ALREADY. OF REMAINING 3 EVEN INTEGERS, CHOOSE 1.)

(9 (ODD) IS IN THE SET ALREADY. OF REMAINING 4 ODD INTEGERS, CHOOSE 3.)

BY INCLUSION/EXCLUSION PRINCIPLE,

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 30 + 35 - 12 = 53$