Discrete Mathematics, MATH 2001-R01

Exam 1 Review

Solutions

1. (a) We are looking for the subsets X of $\{1,2,3\}$ that are subsets of $\{1,2\}$. Since $\{1,2\} \subseteq \{1,2,3\}$, the answer is simply $\mathscr{P}(\{1,2\})$:

$$\{\{1\},\{2\},\{1,2\},\varnothing\}.$$

   (b) We are looking for those sets X such that $X \subseteq \{1,2,3\}$ and $X \subseteq \mathscr{P}(\{1,2\})$. The only possibility is $X = \varnothing$.

2. (a) We have $A \cap B = \{1\}$. Thus

$$(A \cap B) \times A = \{1\} \times \{0,1\} = \{(1,0),(1,1)\}.$$

   (b) We have $\mathscr{P}(A) = \{\varnothing,\{0\},\{1\},\{0,1\}\}$ and $\mathscr{P}(B) = \{\varnothing,\{1\},\{2\},\{1,2\}\}$. Thus

$$\mathscr{P}(A) - \mathscr{P}(B) = \{\{0\},\{0,1\}\}.$$

   (c) $\mathscr{P}(A) \cap \mathscr{P}(B) = \{\varnothing,\{1\}\}$.

   (d) We have $A \cap B = \{1\}$. Thus, $\mathscr{P}(A \cap B) = \{\varnothing,\{1\}\}$.

3. (a) We have

$$\begin{aligned} A-(B \cup C) &= \{x : x \in A \wedge \sim (x \in B \vee x \in C)\} \\ &= \{x : x \in A \wedge (x \notin B \wedge x \notin C)\} = \{x : x \in A \wedge x \notin B \wedge x \notin C\} \\ &= \{x : (x \in A \wedge x \notin B) \wedge \sim (x \in C)\} = \{x : (x \in A - B) \wedge \sim (x \in C)\} = (A-B)-C. \end{aligned}$$

   (b) We have

$$\begin{aligned} A \times (B \cap C) &= \{(x,y) : x \in A \wedge y \in (B \cap C)\} = \{(x,y) : x \in A \wedge y \in B \wedge y \in C\} \\ &= \{(x,y) : x \in A \wedge y \in B \wedge x \in A \wedge y \in C\} \\ &= \{(x,y) : (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C)\} \\ &= \{(x,y) : (x,y) \in A \times B \wedge (x,y) \in B \times C\} = (A \times B) \cap (A \times C). \end{aligned}$$

4. (a) Let P: "this gas has an unpleasant smell", Q: "this gas is explosive" and R: "the gas is hydrogen". Then the statement can be written as $(P \vee \sim Q) \Rightarrow \sim R$.

   (b) Let P: "George has a fever", Q: "George has a headache" and R: "George goes to the doctor". Then the statement can be written as $P \wedge Q \Rightarrow R$.

   (c) Same P, Q and R as in (b). The statement can be written as $P \vee Q \Rightarrow R$.

   (d) $x \neq 2 \wedge x$ prime $\Rightarrow x$ odd.

5. (a) $R: x > 0 \wedge y \leq 0$. $\sim R: x \leq 0 \vee y > 0$.

   (b) $R: x$ prime $\Rightarrow \sqrt{x} \notin \mathbb{Q}$. $\sim R: \exists x$ prime $\wedge \sqrt{x} \in \mathbb{Q}$.

   (c) $R: \forall p$ prime, $\exists q$ prime, $q > p$. $\sim R: \exists p$ prime, $\forall q$ prime, $q \leq p$.

   (d) $\sin(x) < 0 \Rightarrow \sim (0 \leq x \leq \pi)$. $\sim R: \exists x, \sin(x) < 0 \wedge 0 \leq x \leq \pi$.

6. (a) $\forall x \in A, x \in B$.
   Negation: $\exists x \in A, x \notin B$.

   (b) This means $X \subseteq A$: $\forall y \in X, y \in A$.
   Negation: $\exists y \in X, y \notin A$.

   (c) This means: $\forall B \in X, B \in \mathscr{P}(A)$, i.e. $\forall B \in X, B \subseteq A$. Thus we can write:
   $\forall B \in X,\ \forall x \in B,\ x \in A$.
   Negation: $\exists B \in X,\ \exists x \in B,\ x \notin A$.

   (d) We have $X \in \mathscr{P}(A) \wedge X \in \mathscr{P}(B)$, i.e. $X \subseteq A \wedge X \subseteq B$. Thus we can write:
   $(\forall x \in X,\ x \in A) \wedge (\forall x \in X,\ x \in B)$.
   Negation: $(\exists x \in X,\ x \notin A) \vee (\exists x \in X,\ x \notin B)$.

   (e) This means: $\forall X \in \mathscr{P}(A),\ X \in \mathscr{P}(B)$, i.e. $\forall X, X \subseteq A \Longrightarrow X \subseteq B$. We can then write:
   $\forall X, (\forall y \in X,\ y \in A) \Longrightarrow (\forall y \in X,\ y \in B)$.
   Negation: $\exists X,\ (\forall y \in X,\ y \in A) \wedge (\exists y \in X,\ y \notin B)$.

7. (a) Suppose x is even. Then $x = 2k$, for some $k \in \mathbb{Z}$. Thus, $3x+5 = 6k+5 = 2(3k+2)+1$ is odd. Conversely, suppose $3x+5$ is odd. We want to prove that x is even. By contrapositive, suppose x is odd. Thus, $x = 2k+1$, for some $k \in \mathbb{Z}$. Therefore, $3x+5 = 3(2k+1)+5 = 6k+8 = 2(3k+4)$ is even, which proves the statement.

   (b) Let $a \in \mathbb{Z}$ and suppose $14 \mid a$. Then $a = 14k$, for some $k \in \mathbb{Z}$. Thus, $a = 2(7k) = 7(2k)$, meaning that $2 \mid a \wedge 7 \mid a$. Conversely, suppose $2 \mid a \wedge 7 \mid a$. Then $a = 2k \wedge a = 2t$, for some $k, t \in \mathbb{Z}$. It follows that $2k = 7t$, which implies that t must be even. Thus, $t = 2h$, for some $h \in \mathbb{Z}$. Therefore, $a = 7t = 14h$, and $14 \mid a$.

   (c) By contradiction, suppose $\alpha \in \mathbb{Q}$ is a rational solution of $x^3+x+3=0$, i.e. $\alpha^3+\alpha+3=0$. Since $\alpha \in \mathbb{Q}$, we can write $\alpha = \dfrac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p,q) = 1$. We expand and obtain

$$\frac{p^3}{q^3} + \frac{p}{q} + 3 = 0 \Longrightarrow p^3 + pq^2 + 3q^3 = 0.$$

   Notice that p and q cannot be both even, since $\gcd(p,q) = 1$. We then proceed by cases.
   Case 1: p is even and q is odd. Then $p = 2k$ and $q = 2t+1$, for some $k, t \in \mathbb{Z}$. This implies

$$\underbrace{(2k)^3}_{\text{even}} + \underbrace{2k(2t+1)^2}_{\text{even}} + \underbrace{3(2t+1)^3}_{\text{odd}} = 0,$$

   which is a contradiction, since the left side of the equation is an odd number and 0 is even.
   Case 2: p is odd and q is even. Then $p = 2k+1$ and $q = 2t$, for some $k, t \in \mathbb{Z}$. This implies

$$\underbrace{(2k+1)^3}_{\text{odd}} + \underbrace{(2k+1)(2t)^2}_{\text{even}} + \underbrace{3(2t)^3}_{\text{even}} = 0,$$

which is a contradiction, since the left side of the equation is an odd number and 0 is even.

Case 3: p is odd and q is odd. Then $p = 2k+1$ and $q = 2t+1$, for some $k,t \in \mathbb{Z}$. This implies

$$\underbrace{(2k+1)^3}_{\text{odd}} + \underbrace{(2k+1)(2t+1)^2}_{\text{odd}} + \underbrace{3(2t+1)^3}_{\text{odd}} = 0,$$

which is a contradiction, since the left side of the equation is an odd number and 0 is even.

(d) Suppose $a \mid b$ and $a \mid (b^2 - c)$. Then $b = ak$ and $b^2 - c = ta$, for some $k,t \in \mathbb{Z}$. It follows that $(ak)^2 - c = ta \implies a^2k^2 - c = ta \implies a^2k^2 - ta = c \implies a(ak^2 - a) = c$, which implies that $a \mid c$.

(e) By contradiction, suppose $a^2 + b^2 = c^2$, for some $c \in \mathbb{Z}$, and they are both odd. Then $a = 2k+1$ and $b = 2t+1$, for some $k,t \in \mathbb{Z}$. This implies

$$(2k+1)^2 + (2t+1)^2 = c^2 \implies 4k^2 + 4k + 1 + 4t^2 + 4t + 1 = c^2$$

$$\implies 4k^2 + 4t^2 + 4k + 4t + 2 = c^2 \implies c^2 \text{ is even} \implies c \text{ is even.}$$

Therefore, we can write $c = 2p$, for some $p \in \mathbb{Z}$. The above equation then becomes

$$2(2k^2 + 2t^2 + 2k + 2t + 1) = 4p^2 \implies \underbrace{2k^2 + 2t^2 + 2k + 2t + 1}_{\text{odd}} = \underbrace{2p^2}_{\text{even}},$$

which is a contradiction.

(f) We use the Bézout's identity and obtain $1 = as + bt$, for some $s,t \in \mathbb{Z}$. We multiply this equation by c and obtain $c = acs + bct$. Since $a \mid bc$, it follows that $bc = ha$, for some $h \in \mathbb{Z}$. Therefore $c = acs + hat = a(cs + ht)$, which implies that $a \mid c$.

(g) Let $d = \gcd(n, n+1)$. Then $d \mid n \wedge d \mid (n+1)$. This implies $n = kd \wedge n+1 = td$, for some $k,t \in \mathbb{Z}$. Then $kd + 1 = td$, which implies $(t-k)d = 1$, i.e. $d \mid 1$. The only possibility is that $d = 1$ (since $d \geq 0$).

(h) By contradiction, suppose $A \cap (B - A) \neq \varnothing$. Then there exists an element $x \in A \cap (B - A)$. This implies:

$$x \in A \wedge x \in (B - A) \implies x \in A \wedge (x \in B \wedge x \notin A) \implies x \in A \wedge x \notin A,$$

which is a contradiction.

(i) By contradiction, suppose there exist integers a, b such that ab is odd and $a^2 + b^2$ is odd. Then we can write $ab = 2k+1$ and $a^2 + b^2 = 2t+1$, with $k,t \in \mathbb{Z}$. We have

$$(a+b)^2 = a^2 + b^2 + 2ab = 2t + 1 + 2(2k+1) = 2t + 4k + 3,$$

which implies that $(a+b)^2$ is odd, and therefore $a+b$ is odd. This is possible only if a and b have different parity. Without loss of generality, we assume a is even and b is odd. We write $a = 2p$ and $b = 2q+1$, with $p,q \in \mathbb{Z}$. This implies that $ab = 2p(2q+1)$ is even, which is a contradiction.

(j) Suppose $a \equiv b \bmod 10$. Then $a - b = 10k$, for some $k \in \mathbb{Z}$. This implies $a - b = 5(2k) = 2(5k)$, i.e. $a \equiv b \bmod 5$ and $a \equiv b \bmod 2$. Conversely, suppose $a \equiv b \bmod 2$ and $a \equiv b \bmod 5$. Then $a - b = 2k$ and $a - b = 5t$, with $k,t \in \mathbb{Z}$. This implies $2k = 5t$, which is possible only if t is even. Thus, $t = 2h$, with $h \in \mathbb{Z}$, and therefore $a - b = 5(2h) = 10h$, i.e. $a \equiv b \bmod 10$.

(k) By contradiction, suppose there exist $a,b \in \mathbb{Z}$ such that $4 \mid (a^2+b^2)$, and both are odd. Then $a = 2k+1$ and $b = 2t+1$, with $k,t \in \mathbb{Z}$. It follows that $a^2+b^2 = (2k+1)^2+(2t+1)^2 = 4k^2+4k+1+4t^2+4t+1 = 4(k^2+t^2+k+t)+2$, which is not a multiple of 4, which is a contradiction.

(l) By contradiction, suppose $\log_2 3$ is rational, i.e. $\log_2 3 = \frac{a}{b}$, with $a,b \in \mathbb{Z}$, $b \neq 0$, and $\gcd(a,b) = 1$. This implies

$$2^{a/b} = 3 \Longrightarrow 2^a = 3^b,$$

which is a contradiction, since the left side is an even number and the right side is odd.