Exam 2

Answer all 7 questions for a total of 100 points. Write your solutions in the accompanying blue book using complete sentences. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. (15 points) Suppose $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. Prove the statement:
   If $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$, then $c \equiv b \pmod{n}$.

Solution: (Direct proof) Assume that $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$. Then by definition, $n \mid (a-b)$ and $n \mid (a-c)$. Therefore $a - b = nx$ and $a - c = ny$ for some $x, y \in \mathbb{Z}$. Now,

$$c - b = (a-b) - (a-c) = nx - ny = n(x-y).$$

Since $x - y \in \mathbb{Z}$, this shows that $n \mid (c-b)$, and so $c \equiv b \pmod{n}$. □

2. (15 points) Suppose A, B, and C are sets. Prove the statement:
   If $B \subseteq C$, then $A \times B \subseteq A \times C$.

Solution: (Direct proof) Assume $B \subseteq C$ and let $(x, y) \in A \times B$. We must show that $(x, y) \in A \times C$. Since $(x, y) \in A \times B$, we have $x \in A$ and $y \in B$. Since $B \subseteq C$, we have $y \in C$. Thus $x \in A$ and $y \in C$, and so $(x, y) \in A \times C$. □

3. (15 points) Suppose $x, y \in \mathbb{R}$. Prove the statement:
   $(x+y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.

Solution: First, assume that $(x+y)^2 = x^2+y^2$. We must show that $x = 0$ or $y = 0$. By assumption, after expanding $(x+y)^2$ we have

$$x^2 + 2xy + y^2 = x^2 + y^2,$$

and subtracting $x^2 + y^2$ from both sides yields

$$2xy = 0. \tag{1}$$

If $y \neq 0$, then dividing both sides of equation (1) by $2y$ shows $x = 0$. If $x \neq 0$, then dividing both sides of equation (1) by $2x$ shows $y = 0$. The only other possibility is the case when both $x = 0$ and $y = 0$. Regardless of the case, this proves that $x = 0$ or $y = 0$.

Now, to prove the converse, assume that $x = 0$ or $y = 0$. Then $2xy = 0$, and so

$$(x+y)^2 = x^2 + 2xy + y^2 = x^2 + 0 + y^2 = x^2 + y^2.$$

This completes the proof. □

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4. (15 points) Suppose $a, b \in \mathbb{Z}$. Prove the following statement:
If the sum $a + b$ and the product ab are both even, then a and b are both even.
Hint: it is probably easiest to prove the contrapositive statement by cases.

Solution: (Contrapositive) Let us prove the contrapositive statement:

If it is not the case that a and b are both even, then it is not the case that the sum $a + b$ and the product ab are both even.

By De Morgan's law $\sim (P \wedge Q) = (\sim P) \vee (\sim Q)$, an equivalent restatement is the following.

If a is odd or b is odd, then the sum $a + b$ is odd or the product ab is odd.

So assume that at least one of the integers a and b is odd. We break the argument into cases. If exactly one of the integers is odd, say a is odd and b is even, then set $a = 2n + 1$ and $b = 2m$ for some $n, m \in \mathbb{Z}$. Then

$$a + b = (2n + 1) + 2m = 2(n + m) + 1.$$

Since $n + m \in \mathbb{Z}$, this shows that $a + b$ is odd. If both of the integers a and b are odd, then set $a = 2n + 1$ and $b = 2m + 1$ for some $n, m \in \mathbb{Z}$. Then

$$ab = (2n + 1)(2m + 1) = 2nm + 2n + 2m + 1 = 2(nm + n + m) + 1.$$

Since $nm + n + m \in Z$, this shows that ab is odd. In either case, $a + b$ is odd or ab is odd.

5. (15 points) Prove the statement:
$\sqrt[3]{2}$ is irrational.

Solution: (Contradiction) Assume for the sake of contradiction that $\sqrt[3]{2}$ is rational. That is, assume that $\sqrt[3]{2} = a/b$ for some $a, b \in \mathbb{Z}$, $b \neq 0$. Furthermore, assume that a/b is a reduced fraction and, in particular, $\boxed{a \text{ and } b \text{ are not both even}}$.
Now $\sqrt[3]{2} = a/b$, and multiplying both sides by b and cubing both sides produces

$$2b^3 = a^3.$$

Since $b^3 \in \mathbb{Z}$, this shows that a^3 is even. Since the product of odd integers is odd, it must be that $\boxed{a \text{ is even}}$. Set $a = 2n$ for some $n \in \mathbb{Z}$. Now

$$2b^3 = (2n)^3 = 8n^3.$$

Dividing both sides by 2, we have

$$b^3 = 2(2n^3),$$

and since $2n^3 \in \mathbb{Z}$, this shows that b^3 is even. As before, since the product of odd integers is odd, it must be that $\boxed{b \text{ is even}}$ $\Rightarrow\Leftarrow$. This contradicts that fact that a and b are *not* both even. Therefore, our assumption that $\sqrt[3]{2} \in \mathbb{Q}$ must be false, and so $\sqrt[3]{2}$ must be irrational. $\square$

6. (15 points) Prove the statement:
For every positive integer n,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Solution: (Mathematical induction) First observe that the statement is true for $n = 1$.

$$1^3 = \frac{1^2(1+1)^2}{4} \quad \checkmark$$

Now we finish with an inductive argument. Let us assume that the statement is true for n:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4},$$

and use this fact to show that statement is true for $n + 1$:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 = \frac{(n+1)^2((n+1)+1)^2}{4}.$$

We have

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 &= \left(1^3 + 2^3 + 3^3 + \ldots + n^3\right) + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2 + 4(n+1))}{4} \\
&= \frac{(n+1)^2((n+1)+1)^2}{4}.
\end{aligned}$$

This completes the proof by induction. □

7. (10 points) *Disprove* the statement:
For all sets A and B, $\mathscr{P}(A) - \mathscr{P}(B) \subseteq \mathscr{P}(A - B)$.

Solution: (Counterexample) In order to disprove the statement, we prove its negation:

There exist sets A and B such that $\mathscr{P}(A) - \mathscr{P}(B) \nsubseteq \mathscr{P}(A - B)$.

To prove an existence statement like this, it is enough to provide one example (i.e. a *counterexample* to the original statement).

Set $A = \{1, 2\}$ and $B = \{1\}$. Then we have

$$\begin{aligned}
\mathscr{P}(A) &= \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, \\
\mathscr{P}(B) &= \{\emptyset, \{1\}\}, \\
\mathscr{P}(A) - \mathscr{P}(B) &= \{\{2\}, \{1, 2\}\},
\end{aligned}$$

and

$$\mathscr{P}(A - B) = \mathscr{P}(\{2\}) = \{\emptyset, \{2\}\}.$$

Since $A = \{1, 2\}$ is an element of $\mathscr{P}(A)$ but not an element of $\mathscr{P}(A - B)$, this shows that $\mathscr{P}(A) - \mathscr{P}(B) \nsubseteq \mathscr{P}(A - B)$. $\square$