Final Exam Review Problems

Our Final Exam is Thursday 5/11 1:30-3:30 pm and will cover chapters 1-12 in Book of Proof. The following questions are meant to provide an additional opportunity to practice this material efficiently. Solutions are posted to our class website.

1. Analyze the logical form of the following statements, then negate them. You may use the symbols $\in, \notin, =, \neq, \wedge, \vee, \Rightarrow, \Longrightarrow, \forall$ and $\exists$ in your answers, but not $\subseteq, \not\subseteq, \mathcal{P}, \cap, \cup$ or $\sim$.
   (a) $A \subseteq B - C$.
   (b) $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.
2. Negate the following statements.
   (a) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y > x \Longrightarrow \exists z \in \mathbb{R}, y = z^2 + 5z$.
   (b) $\forall a \in A, \exists b \in B, a \in C \Longleftrightarrow b \in C$.
3. Prove the following propositions:
   (a) If $a, b, c \in \mathbb{Z}$, and if $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.
   (b) If $n \in \mathbb{Z}$ is odd, then $8 \mid (n^2 - 1)$.
   (c) Prove that for every integer n, $14 \mid n$ if and only if $2 \mid n$ and $7 \mid n$.
   (d) Prove that it is *not* true that for every integer n, $60 \mid n$ if and only if $6 \mid n$ and $10 \mid n$.
4. Prove by induction that for every integer $n \geq 0$, $4 \mid (3^{2n} - 1)$.
5. Let R be the relation in $\mathbb{Z}$ given by
   $$x \; R \; y \Longleftrightarrow x = y \;\vee\; xy > 0.$$
   (a) Prove that R is an equivalence relation.
   (b) How many equivalence classes does the relation R on $\mathbb{Z}$ have?
6. Let R be a relation on a set A which is reflexive and transitive. Let $S = R \cap R^{-1}$. Prove that S is an equivalence relation on A.
7. Find all the equivalence relations on the set $A = \{1, 2, 3\}$.
8. Let $f : A \to B$ be a function, and let $Y, Z \subseteq B$. Prove the following:
   (a) $f(f^{-1}(Y)) \subseteq Y$.
   (b) $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.
   (c) $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.
9. Show that the function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi(x, y) = (x + y, x - y)$ is bijective and find the inverse function ϕ^{-1}.
10. Let $f : \mathbb{Z} \longrightarrow \mathbb{Q}$ be the function
    $$f(n) = \begin{cases} n^2 - 4 & n \geq 0 \\ \frac{3}{5}n & n < 0 \end{cases}$$
    (a) Prove or disprove that f is injective and/or surjective.
    (b) Prove by induction that, for every $n \geq 1$
    $$\sum_{i=1}^{n} f(i) = \frac{2n^3 + 3n^2 - 23n}{6}.$$
11. Let $A = \{1, 2, 3\}$ and $B = \{\alpha, \beta\}$. Write down all the possible functions $f : A \to B$. How many of those functions are injective? How many are surjective?
12. Let X and Y be two non-empty sets. Let $f : X \to Y$ be a function. We define in X the relation $x \; R \; x'$ if and only if $f(x) = f(x')$. Prove that R is an equivalence relation.

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